# Dynamic Agency and Real Options

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**Abstract**

We analyze how dynamic moral hazard affects corporate investment. In our model, the owners of a firm hold a real option to increase capital. They also employ a manager who controls the firm's productivity, but is subject to moral hazard. Although this conflict reduces capital productivity, both over- and under-investment can occur. When moral hazard is severe, the firm invests at a *lower* threshold in productivity than in the first-best because investment is a substitute for effort. When the growth option is large, the investment threshold is *higher* than in the first-best. We also discuss how investment affects pay-performance sensitivity.

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# 1 Introduction

How firms make real investment decisions is a central topic in the study of corporate finance. As the investments of individual firms are typically lumpy and (partially) irreversible, they are well described as real options. In the standard real options model, cash flows of a firm are generated without any agency conflicts. In reality, cash flow growth often requires managerial effort and when this effort is costly and unobservable, a moral hazard problem arises. We investigate how this moral hazard problem affects investment timing decisions.

In a standard option model, the optimal time to invest is given by the moment at which productivity reaches a threshold such that the benefit of investment equals the direct cost plus the opportunity cost of investment. In our model, moral hazard affects these costs and benefits. On the one hand, for a given level of productivity, moral hazard will decrease the benefit of investment, raising the investment threshold. On the other hand, moral hazard will decrease the opportunity cost of investing, lowering the investment threshold. We show that when the moral hazard problem is severe or the size of the investment option is small, the agency conflict decreases the opportunity cost more than the benefit and hence causes the firm to investment at a lower threshold in productivity – a type of *over-investment*. In contrast, when the moral hazard problem is moderate or the size of the investment option is large, the opposite is true and the agency problem causes the firm to invest at a higher threshold in productivity – a type of *under-investment*.

Empirical evidence indicates that firms often either under- or over-invest relative to some first-best benchmark. For example, Bertrand and Mullainathan (2003) find that when external governance becomes weaker due to the passage of anti-takeover legislation, firms invest less in new plants. In contrast, Blanchard, Lopez-de Silanes, and Shleifer (1994) document that when firms receive an exogenous cash windfall, they increase investment even when they have a low marginal Tobin's $Q$. To explain these findings, two main themes have developed in the theoretical literature on firm investment under agency conflicts. Grenadier and Wang (2005), DeMarzo and Fishman (2007), and DeMarzo, Fishman, He, and Wang (2012) argue that moral hazard in effort induces firms to curtail investment. While Jensen (1986), Stulz (1990), Harris and Raviv (1990), Hart and Moore (1995), and Zwiebel (1996) posit that firms over-invest because managers have a preference for "empire-building." Along similar lines, Roll (1986) and Bernardo and Welch (2001) show that over-investment may occur because managers are over-confident. We show that moral hazard in effort can also cause a form of over-investment; a firm plagued with a severe problem of incentivizing a manager to work will exercise investment options at a lower threshold in productivity than identical firms without a moral hazard problem.

To arrive at this result, we construct a continuous-time dynamic moral hazard model in which an investor contracts a manager to run a firm. This manager can exert effort to increase the growth rate of capital productivity. For example, the manager might need to work to increase market share or improve operational efficiency. This effort is costly to the manager and hidden to the investors so that the manager can potentially gain utility by exerting less effort than would be optimal from the perspective of the investor. In order to incentivize the manager to exert effort, the optimal contract will expose her to firm performance. This exposure is costly because it reduces risk sharing between the manager and the investor. In addition to increasing productivity by contracting with the manager, the investor has an option to irreversibly increase the firm's capital.

An important feature of our model is that capital and managerial effort are complements in the firm's production function. When the manager exerts more effort, productivity increases at a faster rate, and capital becomes more productive. Similarly, when the firm has more capital, managerial effort leads to more growth in cash flow. We view this assumption as consistent with an essential characteristic of a manager's role within a firm, in that managers manage, rather than replace, capital. However, the complementarity of managerial effort and capital in the firm's production function does not necessarily mean the firm will invest less when the price of managerial effort rises. Indeed, if the price of managerial effort rises due to an increase in the cost of incentives, the firm may invest more and decrease incentives. In this sense, investment serves as a substitute for managerial effort as a means of increasing cash flow. It is important to note that this substitution effect is not driven by the manager's preferences for investment. In fact, the manager in our model is indifferent between any particular investment policy.

In addition to implications for the investment behavior of firms, our model also generates results for the manager's compensation and incentives. The power of incentives and pay-performance sensitivity are closely related to the size of the growth opportunity. All else equal, the productivity of managerial effort is increasing in the size of the growth option, and hence so is the power of incentives as measured by the sensitivity of the manager's wealth to firm output, or output-based pay-performance sensitivity. However, increasing the size of the growth option also changes the sensitivity of firm value to output. Consequently, there is a wedge between value-based pay-performance sensitivity and incentives. Value-based pay-performance sensitivity can actually decrease with the size of the growth option. This result is a caveat for empirical work on the power of incentives. In the presence of growth opportunities, there could be a negative relationship between actual incentives and the sensitivity of the manager's wealth to firm value (rather than output).

Our model further predicts that the manager's pay-performance sensitivity can increase or decrease at investment depending on the agency conflicts. When the moral hazard problem is less severe, the optimal contract will call for the manager to exert maximal effort before and after investment. As a result, pay-performance sensitivity, as measured by the sensitivity of the manager's wealth to firm value, will actually increase after investment. If, however, the moral hazard problem is more severe, the optimal contract will call for the manager to significantly decrease effort after investment, which causes a decrease in pay-performance sensitivity. An interesting feature of the results of our model is that the effect of investment on pay-performance sensitivity and the optimal investment policy are closely linked. On the one hand, when investment leads to an increase in pay-performance sensitivity, it must also be the case that an increase the severity of the moral hazard problem raises the investment threshold. On the other hand, when investment leads to a decrease in pay-performance sensitivity an increase the severity of the moral hazard problem lowers the investment threshold.

It is useful to illustrate the model in terms of some real world examples. First, consider a startup firm choosing the optimal time to significantly increase production (for example, during a venture capital funding round or IPO). In this case, the initial capital stock of the firm is small and, as a consequence, managerial effort is relatively cheap. In other words, the start-up manager's moral hazard problem prior to the increase in capital is mild. After increasing capital, the manager's moral hazard becomes more pronounced because she has a larger firm to operate. As a result, increasing the cost of incentives decreases the present value of the added cash flow from the additional capital more than the value of the small firm, causing the firm to wait until productivity has reached a higher threshold before increasing capital. Thus, our model predicts that startup firms with more severe moral hazard problems delay significant expansions. This prediction applies to late stages of venture funding and IPOs.

In the preceding example, the size of the investment option is large relative to the initial capital stock of the firm, thus moral hazard raises the investment threshold. Now consider an example in which the investment opportunity is small relative to the capital stock of the firm. For instance, consider a large mature firm choosing the optimal time to make an acquisition of a small target. In this setting, the acquisition allows the large firm to grow cash flows without providing costly incentives for additional managerial effort. This in turn implies that increasing the cost of incentives has a larger negative effect on the acquiring firm prior to the acquisition than on the merged firm and thus the acquisition. A prediction of the model is then that acquiring firms with more severe agency problems undertake acquisitions at lower thresholds in productivity than they would without the moral hazard problem.

To gain a greater understanding of the forces at work in generating both over- and under-investment, we generalize the model to allow for many different types of investment. The existing models of moral hazard and investment largely consider contracts that implement effort at the first-best level and show that moral hazard decreases or delays investment (e.g., Grenadier and Wang (2005); DeMarzo and Fishman (2007); Biais, Mariotti, Rochet, and Villeneuve (2010); DeMarzo et al. (2012)). We consider a model in the spirit of DeMarzo et al. (2012), i.e., a neoclassical model of investment, in which we allow optimal effort to deviate from the first best. In this case, the marginal value of capital is a sufficient statistic for investment and always decreases with agency problems. Thus, investment decreases with the severity of the moral hazard problem even when effort is flexible. A similar argument applies to a setting with partially irreversible but perfectly divisible investment. When we enrich the model so that the investment technology implies some lumpiness, as is often inherently the case with firm level investment as argued by Doms and Dunne (1998), Caballero and Engel (1999), and Cooper, Haltiwanger, and Power (1999) among others, optimal investment is determined by the average value of new capital. Unlike the marginal value of capital, the average value of capital, and hence the effect of moral hazard on investment, increases or decreases with the severity of the agency problem depending on parameters.

This paper contributes to the growing literature on the intersection of dynamic agency conflicts and investment under uncertainty. On the dynamic contracting side, Holmstrom and Milgrom (1987) and Spear and Srivastava (1987) introduced the notion that providing agents with incentives may take place over many periods. More recently, there has been a renewed interest in dynamic contracting. Biais, Mariotti, Plantin, and Rochet (2007) analyze a rich discrete-time model of a dynamic agency conflict and its continuous-time limit. Much subsequent work builds on the continuous-time approach of Sannikov (2008) to characterize optimal dynamic contracts in a variety of settings. For example, DeMarzo and Sannikov (2006) consider the design of corporate securities when the manager may divert cash. Piskorski and Tchistyi (2010, 2011) derive the optimal design of mortgages when lenders face stochastic interest rates or house prices are stochastic. He (2009) studies optimal executive compensation when firm size follows a geometric Brownian motion. Most closely related to our model of the dynamic agency problem is the capital structure model of He (2011), which allows for a risk-averse agent.

On the investment side, DeMarzo and Fishman (2007), Biais et al. (2010), and DeMarzo et al. (2012) consider dynamic moral hazard with investment. One important distinction between our paper and both Biais et al. (2010) and DeMarzo et al. (2012) is that their setups yield first-best effort even under moral hazard, and as such the substitutability of effort and investment is not present in their models. As consequence moral hazard curtails investment in these models. Szydlowski (2013) studies a capital budgeting problem in which a firm employs a manager who can run multiple projects. In his model, investors sometimes direct the manager to take on negative NPV projects as less inefficient form of punishment than termination.[1]

The investment technology we consider is based on the classic real options models of Brennan and Schwartz (1985) and McDonald and Siegel (1986). Dixit and Pindyck (1994) offer a comprehensive guide to the real options literature. Two papers that use a similar model to ours to evaluate the effects of agency problems on real options investment are Grenadier and Wang (2005) and Philippon and Sannikov (2007). Grenadier and Wang consider a real option exercise problem in the presence of a static moral hazard problem and find that when there is an additional adverse selection over managerial ability, real option exercise is delayed. We consider a dynamic moral hazard problem and find that the real option exercise threshold may either increase or decrease. Philippon and Sannikov consider real options in a dynamic moral hazard setting similar to ours. In their model, cash flows follow an i.i.d process, and as a result, there is no real option problem under the first-best. That is, the firm always immediately invests in the first-best case, as the investment is assumed to be positive net present value. Introducing the agency problem in their setting induces a valuable option to wait to invest until the agent has sufficiently high continuation utility and the firm is very unlikely to be liquidated. Consequently, moral hazard can only delay investment in their setting. In contrast, we model cash flows that grow in expectation and as a consequence optimal managerial effort depends on the level of cash flows and investment and effort may serve as substitutes. This difference means that in our model, unlike in that of Philippon and Sannikov, moral hazard can both raise and lower the investment threshold.

[1] Hirshleifer and Suh (1992) contains a similar result in a static setting. They show that motivating managers to exert effort can also provide them with incentives to take on risky projects that shareholders would reject in the first-best.

There are also a number of papers that consider the effect different type of agency conflict, asymmetric information, on option exercise. For example, Grenadier and Malenko (2011), Morellec and Schürhoff (2011), and Bustamante (2011), and Bouvard (2014) all study how the time of real option exercise can serve as signal of private information.

The rest of the paper proceeds as follows. Section 2 introduces our model of moral hazard and real options. Section 3 provides the optimal contract and investment policy. Section 4 discusses the implications of the moral hazard problem for investment, compensation, and incentives. Section 5 considers a generalization of our basic model to build are greater understanding of the source of the effect of moral hazard on investment. Section 6 concludes.

# 2 The Model

In this section we present our model of dynamic moral hazard and real options. It resembles that of He (2011) in that we consider an agent (the firm's manager) with constant absolute risk-averse (CARA) preferences who can affect the productivity growth of the firm by exerting costly hidden effort. In addition, we endow the firm with an irreversible investment opportunity.

## 2.1 Technology and Preferences

Time is continuous, infinite, and indexed by $t$. The risk-free rate is $r$. A risk-neutral investor employs a risk-averse manager to operate a firm. Firm cash flows are $X_t K_t dt$, where $K_t$ is the level of capital at time $t$ and $X_t$ is a productivity shock with dynamics given by:

$$dX_t = a_t \mu X_t dt + \sigma X_t dZ_t,$$

where $a_t \in [0,1]$ is the manager's effort and $Z_t$ is a standard Brownian motion. Constants $\mu$ and $\sigma$ represent the (net of effort) drift and volatility of the productivity process. Managerial effort here corresponds to any action that increases the growth rate — not the current level — of productivity. For example, the manager may have to exert effort to increase market share or the operational efficiency of the firm. The firm starts with capital $K_0 = k > 0$ and has a one time expansion option to increase capital to $\hat{k}$ at cost $p$. In the notation that follows, a hat indicates a post investment quantity.

The manager has CARA preferences over consumption. She values a stream of consumption $\{c_t\}$ and effort $\{a_t\}$ as:

$$E\left[\int_0^\infty e^{-rt}u(c_t,a_t)dt|\{a\}\right],$$

where $u(c,a) = -e^{-\gamma(c-XKg(a))}/\gamma$ is the manager's instantaneous utility for consumption and effort and $XKg(a)$ is the manager's cost of effort in units of consumption. We assume the manager's normalized cost of effort $g(a)$ is continuously differentiable, increasing, and convex, $g(a) \in \mathcal{C}^1([0,1])$, $g'(a) \geq 0$, $g''(a) > 0$, and $g'(0) = 0$. When we consider specific parameterizations of the model, we assume $g(a)$ is a simple quadratic function. In addition, the manager may save at the risk-free rate $r$. We assume that the manager begins with zero savings. The manager's savings and effort are unobservable to the investor.

The specification for the cost of effort, $XKg(a)$, captures the notion that it is more costly for the manager to increase productivity and cash flows when the firm is larger or more productive. For example, suppose the firm operates factories and that the real option is to build an additional one. On the one hand, the process $X_t$ could represent the volume of goods produced per factory. In this case, managerial effort represents the implementation of a process innovation. While the same such innovation can make identical factories more productive, so that the amount of engineering effort required should not depend on the number of factories run by the firm, implementing that innovation requires the manager to spend a certain amount of time for *each* factory. Thus, increasing the per factory output of two factories is more costly to the manager than of a single factory. On the other hand, the volume of goods produced per factory could be fixed and $X_t$ could represent the profit per unit. The manager could increase profit margins by, for example, increasing demand through marketing. In this case, our specification posits that it is more costly for the manager to increase the profit per unit when the firm produces more units.

## 2.2 Contracts

A contract consists of a compensation rule, a recommended effort level, and an investment policy denoted $\Pi = (\{c_t, a_t\}_{t\geq 0}, \tau)$. The compensation rule $\{c_t\}$ and recommended effort $\{a_t\}$ are stochastic processes adapted to the filtration of public information, $\mathcal{F}_t$. For simplicity, we drop the subscript $t$ whenever we are referring to the entire process of either consumption or effort. The investment policy $\tau$ is $\mathcal{F}_t$-stopping time, which dictates when the firm exercises the option to increase capital. We assume that the investors can directly control investment and will pay the cost of investment. Note that the time $t$ cash flow to the investor under a contract $\Pi$ is given by:

$$dD_t = X_t K_t dt - c_t dt - \mathbb{I}(t = \tau)p,$$

where $D_t$ denotes cumulative cash flow to the investor.

Since the agent can privately save, the compensation, $c_t$, specified by the contract need not be equal to the manager's time $t$ consumption. Denote the manager's accumulated savings by $S_t$ and her actual time $t$ consumption and effort by $\tilde{c}_t$ and $\tilde{a}_t$, respectively. Given a contract $\Pi$, the manager chooses a consumption and effort plan to maximize her utility from the contract:

$$\begin{aligned} W(\Pi) = \max_{\{\tilde{c},\tilde{a}\}} E\left[\int_0^\infty -\frac{1}{\gamma}e^{-\gamma(\tilde{c}_t - X_t K_t g(\tilde{a}_t))-rt}dt\right] \\ \text{such that} \quad dS_t = rS_t dt + (c_t - \tilde{c}_t)dt, S_0 = 0 \\ dX_t = \tilde{a}_t \mu X_t dt + \sigma X_t dZ_t \\ K_t = k + (\hat{k} - k)\mathbb{I}(t \geq \tau). \end{aligned} \tag{1}$$

The dynamics of savings $S_t$ reflect that the difference between compensation $c_t$ and consumption $\tilde{c}_t$ goes to increase (or decrease) savings while the balance grows at the risk-free rate $r$. In addition to the dynamics for $S_t$ given above, we impose the standard transversality condition on the consumption process. The dynamics of productivity, $X_t$, reflect that the expected growth rate of productivity depends on the actual effort, $\tilde{a}_t$, of the manager. Finally, the time $t$ capital stock of the firm depends on the investment policy set forth in the contract.

Given an initial outside option of the manager $w_0$, the investor then solves the problem:

$$\begin{aligned} B(X_0, w_0) = \max_{\{c,a\},\tau} E\left[\int_0^\infty e^{-rt}dD_t\right] \\ \text{such that} \quad dX_t = \tilde{a}_t \mu X_t dt + \sigma X_t dZ_t \\ K_t = k + (\hat{k} - k)\mathbb{I}(t \geq \tau) \\ w_0 \leq E^{(\{\tilde{c},\tilde{a}\},\tau)}\left[\int_0^\infty -\frac{1}{\gamma}e^{-\gamma(\tilde{c}_t - X_t K_t g(\tilde{a}_t)-rt}dt\right], \end{aligned} \tag{2}$$

where $\{\tilde{c}, \tilde{a}\}$ solves problem (1).

We call a contract $\Pi$ *incentive compatible* and *zero savings* if the solutions $\{\tilde{c}_t\}$ and $\{\tilde{a}_t\}$ to Problem (1) are equal to the payment rule and recommended effort plan given in the contract. As is standard in the literature, we focus on contracts in which the solution to problem (1) is to follow the recommended action level and maintain zero savings by virtue of the following revelation-principle result.

**Lemma 1.** *For an arbitrary contract $\tilde{\Pi}$, there is an incentive compatible and zero-savings contract $\Pi$ that delivers at least as much value to the investor.*

# 3 Solution

The solution follows the now standard martingale representation approach developed by Sannikov (2008). The first step is to give a necessary and sufficient condition for a contract to implement zero savings. We then represent the dynamics of the manager's continuation utility (the expected present value of her entire path of consumption) as the sum of a deterministic drift component and some exposure to the unexpected part of productivity growth shocks via the martingale representation theorem. With these dynamics in hand, we characterize the incentive compatibility condition as a restriction on the dynamics of continuation utility. Given the dynamics of continuation utility and productivity implied by incentive compatibility, we can represent the investor's optimal contracting problem as a dynamic program resulting in a system of ordinary differential equations (ODEs) for investor value together with boundary conditions that determine the investment policy. In the Appendix, we provide verification that the solution to this system of ODEs indeed achieves the optimum investor value.

## 3.1 The No-savings Condition

In this subsection, we follow He (2011) to characterize a necessary and sufficient condition for the manager to choose consumption equal to her compensation and thus maintain zero savings. In words, the condition states that the manager's marginal utility for consumption is equal to her marginal utility for savings. To determine the manager's marginal utility for an additional unit of savings, we first consider the impact of an increase in savings on her optimal consumption and effort plan going forward. Suppose $\{\tilde{c}, \tilde{a}\}$ solves problem (1) for a given contract that implements zero savings. Now suppose we simply endow the manager with savings $\mathcal{S} > 0$ at some time $t > 0$. How would her consumption and effort plan respond? Due to the absence of wealth affects implied by the manager's CARA preferences, the optimal consumption plan for $s \geq t$ would be just $\tilde{c}_s + r\mathcal{S}$, while the effort plan would remain unchanged. Thus, an increase in savings from zero to $\mathcal{S}$ increases the manager's utility flow by a factor of $e^{-\gamma r S}$ forever.[2] To make this intuition formal, it is useful to define the manager's continuation utility for a given contract when following the recommended effort policy and accumulating savings $\mathcal{S}$ up to time $t$,

$$W_t(\Pi, \{X_s, K_s\}_{s \leq t}; \mathcal{S}) = \max_{\{\tilde{c}_t, \tilde{a}_t\}} E\left[\int_t^\infty -\frac{1}{\gamma} e^{-\gamma(\tilde{c}_s - X_s K_s g(\tilde{a}_t)) - r(s-t)} ds | \{X_s, K_s\}\right] \tag{3}$$

$$\text{such that} \quad dS_s = rS_s ds + (\tilde{c}_s - c_s)ds \quad S_t = \mathcal{S}$$

$$dX_s = \tilde{a}_s \mu X_s ds + \sigma X_s dZ_s$$

$$K_s = k + (\hat{k} - k)\mathbb{I}(s \geq \tau).$$

The definition of continuation utility and the intuition given above lead to Lemma 2.

[2]Since utility is always negative, the factor $e^{-\gamma r S} < 1$ represents an increase in utility.

**Lemma 2** (He (2011))**.** *Let $W_t(\Pi, \{X_s, K_s\}_{s\leq t}; \mathcal{S})$ be the solution to problem (3), then:*

$$W_t(\Pi, \{X_s, K_s\}_{s\leq t}; S) = e^{-\gamma r S} W_t(\Pi, \{X_s, K_s\}_{s\leq t}; 0). \tag{4}$$

Equation (4) allows us to determine the manager's marginal utility for savings under a contract that implements zero savings:

$$\frac{\partial}{\partial S} W_t(\Pi, \{X_s, K_s\}_{s\leq t}; S)\Big|_{S=0} = -\gamma r W_t(\Pi, \{X_s, K_s\}_{s\leq t}; 0). \tag{5}$$

Since we focus on zero-savings contracts, from now on we drop the arguments and refer simply to continuation utility $W_t$. For the manager to maintain zero savings, her marginal utility of consumption must be equal to her marginal utility of savings:

$$u_c(c_t, a_t) = -\gamma r W_t$$

which, together with the CARA form of the utility function, implies the convenient no-savings condition:

$$u(c_t, a_t) = r W_t. \tag{6}$$

Thus, for a contract to implement zero savings, the manager's flow of utility from the contract must be equal to the risk-free rate $r$ times her continuation utility. This is intuitive; in order for the manager to have no incentive to save, the contrast must deliver the risk-free yield of her continuation utility in units of utility flow. For the remainder of the paper, we only consider contracts that satisfy the no-savings condition given by Equation (6).

## 3.2 Incentive Compatibility

Now that we have characterized a necessary and sufficient condition for a contract to implement zero savings, we turn our attention to the incentive compatibility condition. For an arbitrary incentive compatible and zero-savings contract, consider the following process:

$$F_t = E_t \left[ \int_0^\infty e^{-rs} u(c_s, a_s) ds \right].$$

This process is clearly a martingale with respect to the filtration of public information $\mathcal{F}_t$, thus the martingale representation theorem implies that there exists a progressively measurable process $\beta_t$ such that:

$$dF_t = \beta_t(-\gamma r W_t) e^{-rt} \left( dX_t - a_t \mu X_t dt \right). \tag{7}$$

Now note that that $F_t$ is related to the manager's continuation utility $W_t$ (under the recommended consumption and effort plan) by:

$$dW_t = (rW_t - u(c_t, a_t))dt + e^{rt} dF_t. \tag{8}$$

Combining the no-savings condition (6) with Equations (7) and (8) gives the following dynamics for the manager's continuation utility:

$$dW_t = \beta_t(-\gamma r W_t) \left( dX_t - a_t \mu X_t dt \right). \tag{9}$$

The process $\beta_t$ is the sensitivity of the manager's continuation utility to unexpected shocks to the firm's productivity. Since a deviation from the recommended effort policy results in an unexpected (from the investor's perspective) shock to productivity, $\beta_t$ measures the manager's incentives to deviate from the contract's recommended effort policy.

For a given contract, Problem (1) implies that the manager chooses her current effort to maximize the sum of her instantaneous utility, $u(c_t, a_t)dt$, and the expected change in her continuation utility, $W_t$. The manager's expected change in continuation utility from deviating from the recommended effort policy $a_t$ to $\tilde{a}_t$ is:

$$E[dW_t|\tilde{a}] = \beta_t(-\gamma r W_t)(\tilde{a} - a_t)\mu X_t dt.$$

Thus, incentive compatibility requires that:

$$a_t = \arg\max_{\tilde{a}} \{u(c_t, \tilde{a}) + \beta_t(-\gamma r W_t)(\tilde{a} - a_t)\mu X_t\}. \tag{10}$$

Taking a first order condition for Problem (10) yields:

$$u_a(c_t, a_t) + \beta_t(-\gamma r W_t)\mu X_t = 0.$$

It is straightforward to show that this is a necessary and sufficient condition for the manager's optimal effort plan. Note that $u_a(c_t, a_t) = -u_c(c_t, a_t)X_t K_t g'(a)$ and recall that the no-savings condition is $u_c(c_t, a_t) = (-\gamma r W_t)$, so that we can solve the first order condition above to find:

$$\beta_t = \frac{1}{\mu} K_t g'(a_t). \tag{11}$$

Intuitively, the sensitivity, $\beta_t$, that is required for incentive compatibility is the agent's marginal cost of effort, $X_t K_t g'(a_t)$, scaled by the marginal impact of effort on output, $\mu X_t$. Lemma 3 characterizes incentive-compatible no-savings contracts.

**Lemma 3.** *A contract is incentive compatible and has no savings if and only if the solution $W_t$ to Problem (3) has dynamics given by Equation (9), where $\beta_t$ is defined by Equation (11).*

It is useful to represent the agent's continuation utility, $W_t$, in terms of its certainty equivalent, $V_t = -1/(\gamma r)\ln(-\gamma r W_t)$. Applying Ito's lemma to (9) and combining it with Lemma 3 yields that the dynamics of $V_t$ under an incentive-compatible no-savings contract are given by:

$$dV_t = \frac{1}{2}\gamma r \left(\frac{\sigma}{\mu} X_t K_t g'(a_t)\right)^2 dt + \frac{\sigma}{\mu} X_t K_t g'(a_t) dZ_t. \tag{12}$$

The drift term in Equation (12) comes from the difference in risk aversion between the investor and the manager. Since the manager is risk averse, the certainty equivalent of $W$ must have additional drift for each additional unit of volatility. Since $W$ is a martingale, the drift term in $V$ is entirely due to this effect. This positive drift will show up in the investor's Hamilton-Jacobi-Bellman (HJB) equation as the cost of providing incentives.

## 3.3 First Best

As a benchmark, we first solve the model assuming effort is observable so that there are no agency conflicts. In this case, the investor simply pays the manager her cost of effort. Additionally, if the agent's promised utility is $W$, its certainty equivalent, $V$, can be paid out immediately. Thus, the investor's first-best value function, $B_{FB}$, depends linearly on the certainly equivalent of $W$, or $B_{FB}(X, V) = b_{FB}(X) - V$, for some function $b_{FB}(X)$. We refer to this function as the investor's gross value to indicate that it is equal to the investor's value gross of the certainty equivalent owed to the manager. To solve the for the investor's first-best value function, we simply maximize the value of cash flows from the firm less the (direct) cost of effort. The investor's post-investment gross value function, $\hat{b}_{FB}$, then solves the following HJB equation:

$$r\hat{b}_{FB} = \max_{a\in[0,1]} \left\{ X\hat{k}(1-g(a)) + a\mu X\hat{b}'_{FB} + \frac{1}{2}\sigma^2 X^2 \hat{b}''_{FB} \right\}. \tag{13}$$

Recall that a hat refers to a post-investment quantity. The first two terms in the brackets in Equation (13) are instantaneous cash flows and the cost of effort, respectively, and the other two terms reflect the impact of the dynamics of $X$ on the value function. As all flows are proportional to $X$, the solution is also expected to be linear in $X$ and as a result the optimal effort level given will be constant in $X$. We can solve Equation (13) to find the investor's first-best gross value function:

$$\hat{b}_{FB}(X) = \frac{1-g(\hat{a}^{FB})}{r-\hat{a}^{FB}\mu} X\hat{k}.$$

Before investment, the firm's cash flows and the cost of effort are proportional to the lower level of capital, $k$. The HJB equation for the pre-investment gross firm value, $b_{FB}(X)$, is thus:

$$rb_{FB} = \max_{a\in[0,1]} \left\{ Xk(1-g(a)) + a\mu X b'_{FB} + \frac{1}{2}\sigma^2 X^2 b''_{FB} \right\}. \tag{14}$$

Note that $b'_{FB}$ is not constant due to the curvature implied by the option to invest. Consequently, optimal effort prior to investment, $a^{FB}$, will not necessarily be constant in $X$. To solve the first-best gross firm value prior to investment, we must identify a set of boundary conditions in addition to the HJB equation. At a sufficiently high level of $X$, denoted by $\overline{X}_{FB}$, the firm pays the investment cost $p$ to increase the capital to $\hat{k}$. The firm value at

$\overline{X}_{FB}$ must satisfy the usual value-matching and smooth-pasting conditions:

$$b_{FB}(\overline{X}_{FB}) = \hat{b}_{FB}(\overline{X}_{FB}) - p,$$
$$b'_{FB}(\overline{X}_{FB}) = \hat{b}'_{FB}(\overline{X}_{FB}).$$

Additionally, the firm value is equal to zero as $X$ reaches its absorbing state of zero:

$$b_{FB}(0) = 0.$$

### 3.4 Optimal Contracting and Investment

We now present a heuristic derivation of the optimal contract in the full moral hazard case. First we characterize the payment rule to the manager. Recall that the no-savings condition in Equation (6) provides a link between instantaneous utility and continuation utility. This allows us to express the manager's compensation as a function of the current state of the firm $(X_t, K_t)$, the recommended effort level $a_t$, and the certainty equivalent of her continuation utility $W_t$ as follows:

$$c_t = X_t K_t g(a_t) + rV_t. \tag{15}$$

The first term in Equation (15) is the manager's cost of effort in consumption units, while the second is the risk-free rate times the certainty equivalent of her continuation utility. In other words, the contract pays the manager her cost of effort plus the yield on her continuation utility.

The next task is to calculate the value of the firm to the investor before and after the investment. This amounts to expressing the investor's optimization problem given in (2) as a system of HJB equations. First, we consider the investor's problem post investment. An application of Ito's formula plus the dynamics of $X_t$ and $V_t$ yields the following HJB equation for the value function $\hat{B}$ post investment:

$$r\hat{B} = \max_{a\in[0,1]} \Bigg\{ X\hat{k}(1-g(a)) - rV + a\mu X\hat{B}_X + \frac{1}{2}\sigma^2 X^2 \hat{B}_{XX} \\ + \frac{1}{2}\gamma r \left(\frac{\sigma}{\mu} X_t \hat{k} g'(a)\right)^2 \hat{B}_V + \frac{\sigma^2}{\mu} X^2 \hat{k} g'(a) \hat{B}_{XV} + \frac{1}{2}\left(\frac{\sigma}{\mu} X\hat{k} g'(a)\right)^2 \hat{B}_{VV} \Bigg\}. \tag{16}$$

We guess that $\hat{B}(X,V) = \hat{b}(X) - V$. Again, we refer to $\hat{b}(X)$ as the investor's gross firm value as it measures the investor's valuation of the firm gross of the certainty equivalent promised to the manager. Then $\hat{B}_V = -1$, $\hat{B}_{XV} = 0$, and $\hat{B}_{VV} = 0$. This leaves the following HJB equation for $\hat{b}(X)$:

$$r\hat{b} = \max_{a\in[0,1]} \left\{ \hat{h}(X,a) + a\mu X\hat{b}' + \frac{1}{2}\sigma^2 X^2 \hat{b}'' \right\}, \tag{17}$$

where:

$$\hat{h}(X,a) = X\hat{k}(1-g(a)) - \frac{1}{2}\gamma r \left(\frac{\sigma}{\mu} X\hat{k} g'(a)\right)^2 \tag{18}$$

is the total cash flow to the firm net of effort and incentive costs. It is instructive to note the difference between Equations (13) and (17). In the first-best case, the investor only needs to compensate the manager for her cost of effort, while in the moral hazard case, the investor must also bear incentive costs given by the second term in $\hat{h}$. These are costs for the risk-neutral investor of providing incentives to a risk-averse agent. The incentive cost of effort is proportional to the square of the level of cash flows, $X\hat{k}$, and thus the value function

$\hat{b}(X)$ is no longer linear in $X$ as in the first-best case. It is left to specify the boundary conditions that determine a solution to ODE (17). The first boundary condition is that the firm, gross of the consumption claim to the agent, must be valueless when productivity is zero as this is an absorbing state:

$$\hat{b}(0) = 0. \tag{19}$$

The second boundary condition obtains by noting that the cost of positive effort goes to infinity as $X$ goes to infinity, and as a result the optimal effort goes to zero. Thus, the value function must approach a linear function consistent with zero effort as $X$ goes to infinity:

$$\lim_{X\to\infty} \left| \hat{b}'(X) - \frac{\hat{k}}{r} \right| = 0. \tag{20}$$

We now turn to the pre-investment firm value $B$. We again guess that $B(X, V) = b(X) - V$, where $b$ is the investor's firm value gross of the certainty equivalent promised to the manager. A similar argument to the above leads to the HJB equation for $b$:

$$rb = \max_{a\in[0,1]} \left\{ h(X, a) + a\mu X b' + \frac{1}{2}\sigma^2 X^2 b'' \right\}, \tag{21}$$

where $h(X, a)$ is as $\hat{h}(X, a)$ in Equation (18) with $\hat{k}$ replaced by $k$. Equation (21) is similar to the post-investment ODE given by Equation (17) but for the level of employed capital. A solution to ODE (21) is determined by investment-specific boundary conditions. As in the first-best case, the optimal investment policy will be a threshold $\overline{X}$ in productivity at which the investor will increase the capital of the firm. Again the value-matching and smooth-pasting conditions apply[3]:

$$b(\overline{X}) = \hat{b}(\overline{X}) - p \tag{22}$$

$$b'(\overline{X}) = \hat{b}'(\overline{X}). \tag{23}$$

Additionally, as $X$ reaches zero, the gross firm value is zero:

$$b(0) = 0. \tag{24}$$

We collect our results on the optimal contract in Proposition 1.

**Proposition 1.** *The optimal contract is given by the payment rule (15) and investment time* $\tau = \min\{t : X_t \geq \overline{X}\}$ *such that the investor's gross firm value before and after investment,* $b$ *and* $\hat{b}$*, solve (21)-(24) and (17)-(20).*

Note that our choice to endow the manager with CARA preferences and the ability to privately save allows us to additively separate the dependence of the investor's value on productivity $X_t$ and the certainty equivalent of the manager's continuation utility $V_t$. As a result, the investment problem reduces to the ODE in (21)-(24). If we had considered a risk-neutral manager, the resulting investment problem would be substantially more complex, with two state variables and the optimal investment threshold as a curve in $(X_t, W_t)$ space.

[3]These conditions derive from the usual value-matching and smooth-pasting conditions on the investor's value function $B$: $B(\overline{X}, V) = \hat{B}(\overline{X}, V) - p$ and $B_X(\overline{X}, V) = \hat{B}_X(\overline{X}, V)$.

# 4 Implications for Investment, Compensation, and Incentives

In this section, we discuss the implications of the optimal contract characterized in Proposition 1 for investment, compensation, and incentives. In numerical illustrations of these implications, we use particular parameterizations of our model and the following function form for the normalized cost of effort:

$$g(a) = \frac{1}{2}\theta a^2. \tag{25}$$

Following He (2011), we use a risk-free rate of $r = 5\%$ and a standard deviation of productivity growth of $\sigma = 0.25$. We choose a slightly lower upper bound on the growth rate of productivity of $\mu = 4\%$, which reflects the that in our model the growth rate of productivity is bounded below by 0 due to the non-negativity of effort and the multiplicative specification for the effect of effort on productivity, while some calibrations (e.g., Goldstein, Ju, and Leland (2001)) find negative average growth rates. The parameter of risk aversion $\gamma$ is set equal to 1. Investment increases capital from $k = 0.5$ to $\hat{k} = 1$ at cost 10 per unit of new capital. The cost of effort parameter is $\theta = 1$. We choose parameters for the cost of effort and investment so that the two are close substitutes.

## 4.1 Investment and Moral Hazard

In contrast to the extant literature, we find that moral hazard can increase investment. The key intuition is that effort and investment are (imperfect) substitutes.[4] One period

[4] The substitutability of effort and investment was first emphasized in Holmstrom and Weiss (1985).

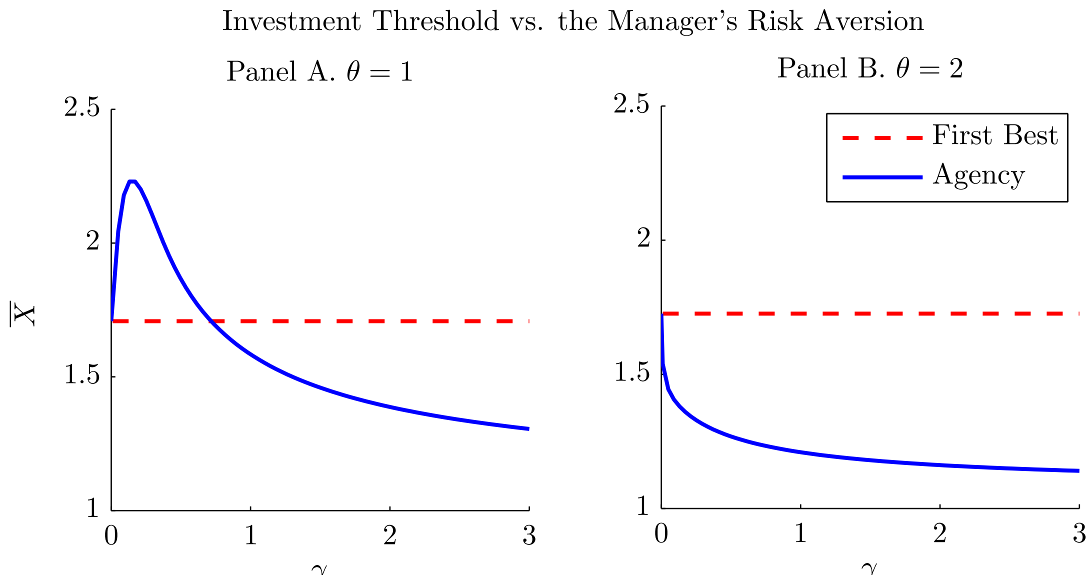


**Figure 1.** The investment threshold $\overline{X}$ as a function of the manager's risk aversion.

of high effort leads to one period of high expected cash flows growth. In a similar way, an investment in additional capital increases cash flows. A key difference between these methods of increasing cash flow growth is that effort is unobservable while investment is contractable. Thus, the relative cost of these two technologies depends on the severity of the moral hazard problem. Intuitively, when the moral hazard problem is severe, investment is a relatively cheap way of growing cash flows. Figures 1-3 show the investment threshold for the moral hazard and first-best cases over a range of parameter values. When the cost of effort $\theta$, the manager's risk aversion $\gamma$, or the size of the investment option is large, the moral hazard problem is less severe. In this case, higher effort is not too costly to implement and the investment threshold is higher for the moral hazard case than for the first best. In contrast, when any of these parameters are high, implementing high effort is costly relative to investment and the investment threshold for the moral hazard case is below that of the first-best case.

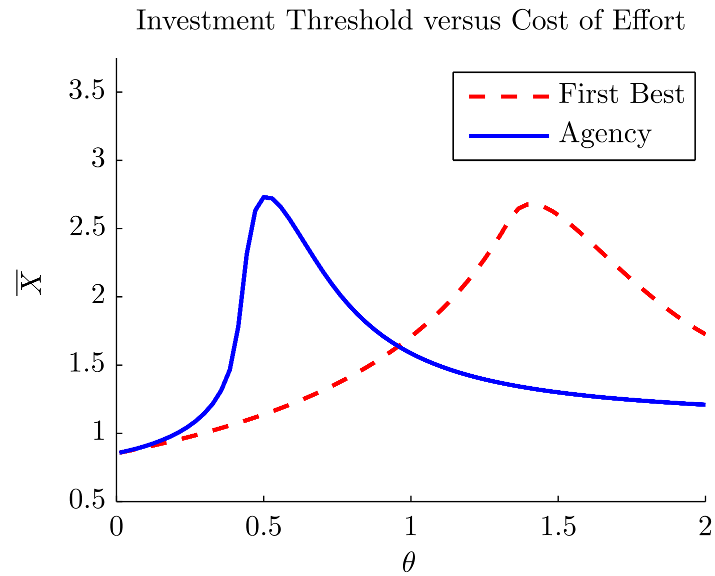


**Figure 2.** The investment threshold $\overline{X}$ as a function of the manager's cost of effort. For low costs of effort, the investment threshold in the moral hazard setting is above that of the first best. For high costs of effort, the relationship is reversed.

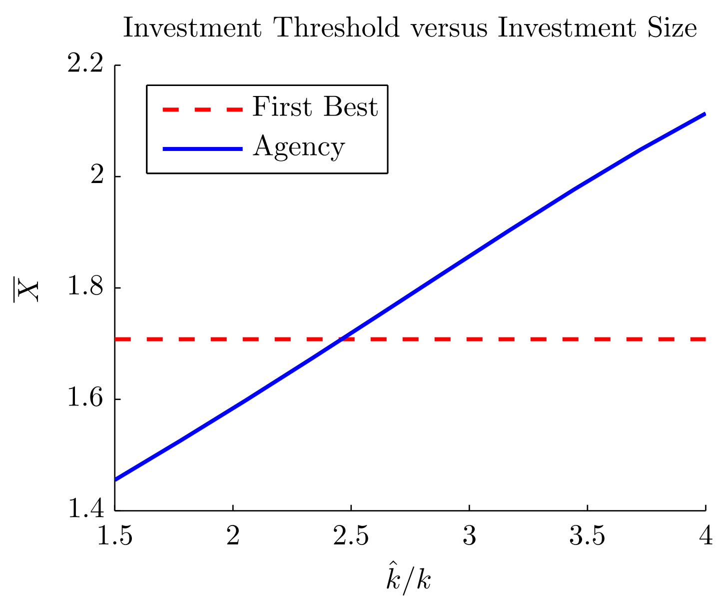


**Figure 3.** The investment threshold $\overline{X}$ as a function of the size of investment.

In order to make this intuition precise, we examine the comparative static properties of firm value before and after investment. Specifically, we consider the following comparative static:

$$\frac{\partial}{\partial \gamma}\left[\hat{b}(X) - b(X)\right]$$

for $X$ close to $\overline{X}$. If this comparative static is negative, then an increase in $\gamma$ decreases the difference between the firm's value before and after investment. In other words, investment is less attractive and the investment threshold will increase. However, when this comparative static is positive, an increase in $\gamma$ increases the profitability of investment and the investment threshold decreases. To compute the derivative above, we apply the method of comparative statics developed by DeMarzo and Sannikov (2006). The details of this derivation are given in the Appendix. The main intuition is that for $X$ very close to the investment boundary, the difference between the pre- and post-investment firm net of the cost of new capital is essentially just the difference between cash flows over the final instant before investment. We can then differentiate cash flow with respect to $\gamma$ to get:

$$\frac{\partial}{\partial \gamma}\left[\hat{b}(X) - b(X)\right] \approx \frac{1}{2} r \left(\frac{\sigma}{\mu}\theta X\right)^2 ((kg'(a^*))^2 - (\hat{k}g'(\hat{a}^*))^2). \tag{26}$$

When the right-hand side of Equation (26) is positive, a small increase in the manager's risk aversion $\gamma$ leads to an increase in the difference between $\hat{b}(\overline{X})$ and $b(\overline{X})$. By the value-matching condition, this means that the investment threshold must decrease. We formally state this result in Proposition 2.

**Proposition 2.** *The investment threshold $\overline{X}$ decreases in $\gamma$ if the marginal cost of effort at* *the optimum drops by a sufficiently large amount at investment, i.e., if*

$$\hat{k}g'(\hat{a}^*(\overline{X})) \leq kg'(a^*(\overline{X})), \tag{27}$$

*otherwise it increases.*

Proposition 2 highlights one of our main findings: increased moral hazed problems do not necessarily lead to decreased investment. In fact, in our model, an increase in managerial risk aversion can lead to a decrease in the investment threshold. This result is driven by the fact that although moral hazard decreases the value of the firm after investment, it also decreases the value of not investing. Since the investment decision is driven by the difference between firm value with and without increased capital, in other words the average value of new capital, moral hazard can decrease the investment threshold. The intuition is that when the moral hazard problem becomes more severe, the investors optimally demand less managerial effort when the firm employs more capital. As a result, the sensitivity of firm value to the incentive cost of effort is less negative for the larger firm.

In light of the condition given in Proposition 2, we now return to a discussion of Figures 1-3. In Panel A of Figure 1, the investment threshold decreases in $\gamma$ once $\gamma$ is large. In this case, a larger $\gamma$ implies a higher cost of incentives, leading to a larger decrease in effort post investment. Similar intuition applies to Figure 2 with the cost of effort coefficient $\theta$. In Panel B of Figure 1, $\theta$ is large enough so that any increase in $\gamma$ leads to a larger reduction in effort post investment and the investment threshold is decreasing in $\gamma$ for all $\gamma$. Finally, when the investment opportunity leads to a the large change in capital, as in the right side of Figure 3, the marginal cost of incentives increases post investment, and Proposition 2 implies that the investment threshold for the agency case will be above that of first-best.

The preceding discussion shows that depending on parameter values, investment under moral hazard can occur at lower or higher levels of productivity compared to the first-best case. Given that the dynamics of the firm under moral hazard and first-best are different, a lower threshold does not necessarily lead to earlier investment when measured in time. Similarly, a higher threshold does not necessarily mean a later investment. To analyze the effect of the agency conflicts on the timing of investment, we now analyze the expected time to invest. Let $\tau^*$ denote the stopping time to reach the optimal investment threshold $\overline{X}$ under the dynamics of productivity $X$ with equilibrium effort $a^*$. Then the expected time to invest starting from $X_0$ is defined as $E[\tau^* \,|X_0]$.

Figure 4 presents a comparison of the expect time to invest under moral hazard and the first-best case over a range of the manager's risk aversion, $\gamma$, and the initial level of productivity, $X_0$. In the regions labeled $D$, investment is delayed by moral hazard; in the regions labeled $A$, investment is accelerated. Whenever the threshold on productivity under moral hazard is above the first-best threshold, i.e., for $\gamma$ lower than 0.7, the investment is delayed by moral hazard. Investment is always delayed in this region because the optimal effort under moral hazard is lower than under the first-best leading to lower expected productivity growth. In the region with high $\gamma$, investment under moral hazard occurs at a lower threshold than under the first-best. As a result, investment is accelerated under moral hazard if the starting productivity $X_0$ is not too low. However, if $X_0$ is sufficiently low, then the decreased optimal effort under moral hazard dominates the lower threshold, and investment is delayed. Similar qualitative results obtain for other varying parameter values, such as $\theta$ or $\hat{k}/k$.

The characteristics of the expected time to investment confirm our interpretation of the investment threshold as a measure of over- or under-investment. When the threshold is above

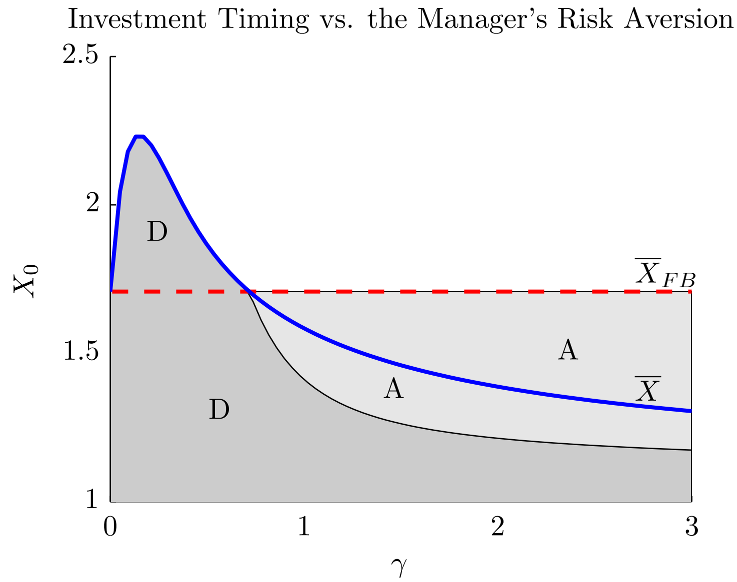


**Figure 4.** The expected time to invest relative to the first-best expected time to invest as a function of the manager's risk aversion $\gamma$ and initial level of productivity $X_0$. D denotes delayed and A accelerated investment relative to the first-best.

that of the first-best, what we call under-investment, the firm also under-invests in terms of timing. When the threshold is below that of the first-best, what we call over-investment, the firm over-invests in terms of timing for higher levels of initial productivity.

## 4.2 Incentives and Pay-Performance Sensitivity

In this section, we consider the implications of real options for managerial incentives. Specifically, we examine the effect of the presence and size of growth options on the measurement of pay-performance sensitivity (PPS) as well its behavior at the moment the firm exercises the growth option. While there has been a robust empirical investigation into the relation between PPS and firm size, see Murphy (1999) and Frydman and Jenter (2010) for a review of this literature, there has been less attention paid to the relation between endogenous investment and PPS. Our results provide guidance for future empirical work on this direction.

The manager's compensation and incentives depend on the level of effort stipulated by the optimal contract. Therefore, we begin our inquiry with a discussion of managerial effort. For interior solutions of effort $a$, we use the HJB equations (17) and (21) to characterize the optimal effort policies $a^*(X)$ and $\hat{a}^*(X)$ by the first-order conditions:

$$g'(a^*(X)) = \frac{\mu^3 b'(X)}{k(\mu^2 + \gamma r \sigma^2 X k g''(a^*(X))}, \tag{28}$$

$$g'(\hat{a}^*(X)) = \frac{\mu^3 \hat{b}'(X)}{\hat{k}(\mu^2 + \gamma r \sigma^2 X \hat{k} g''(\hat{a}^*(X))}. \tag{29}$$

In the following analysis, we restrict our attention to parameter values such that the maxima $a^*(X)$ and $\hat{a}^*(X)$ satisfy the second-order conditions.[5] Optimal effort is time-varying with productivity $X_t$, depends on the primitive parameters of the model, and on the presence of growth opportunities. Figure 5 illustrates some of the key properties of the optimal effort for our baseline parameter values. Efforts in young (pre-investment), mature (post-investment), and small no-growth (permanently small) firms are plotted at two levels of the cost of effort, $\theta$. Effort implemented in the mature and no-growth firms decreases and goes to zero as $X$ approaches infinity. This is because the cost of providing incentives grows more in $X$ than does the benefit of effort. A related effect makes effort decrease in response to exogenous changes in capital (that is, abstracting from growth options; to see this, compare the efforts of the no-growth and mature firms).

Effort implemented in the young firm is above that of the mature firm due to two reasons. First, the young firm employs a low level of capital. This property also manifests itself in

[5] If the second-order derivative of the objective function is zero (a knife-edge case given its dependence of $X$), then the implicit function theorem is not applicable.

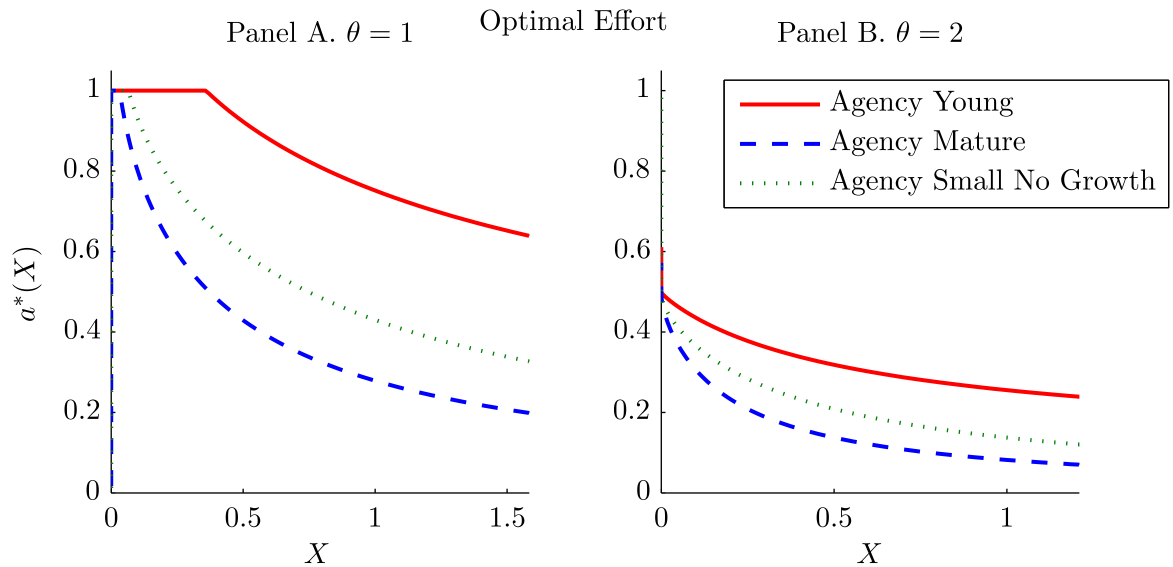


**Figure 5.** The optimal effort at two levels of the cost of effort: $\theta = 1$ (left) and $\theta = 2$ (right).

the fact that effort (weakly) decreases at the moment of investment. The second reason for high effort in young firms is due to the presence of growth options. As is standard in real options models, growth options increase the sensitivity of firm value to productivity shocks as the firm approaches the investment threshold. As the optimal effort increases in $b'(X)$ (see Equation (28)), this indicates that effort may increase in $X$ in the young firm. Intuitively, the prospect of capital investment makes contracting high effort additionally attractive from the investor's point of view.

To implement any of the optimal effort levels under moral hazard, the manager needs to be appropriately incentivized. To determine how investment opportunities affect the power of incentives, we look at two alternative measures thereof: one implied by our model and another commonly used in practice. A direct measure of a manager's incentives in our model is the sensitivity of her dollar (certainty-equivalent) continuation utility to productivity shocks.[6] Prior to investment, the optimal contract sets this quantity to

$$\beta^*(X) = \frac{1}{\mu} k g'(a^*(X)) \tag{30}$$

with an equivalent formula for post-investment incentives $\hat{\beta}^*(X)$. Note that this expression follows directly from substituting the optimal effort policy $a^*(X)$ into the incentive compatibility condition given by Equation (11).

A standard approach to the measurement of pay-performance sensitivity is to compute the sensitivity of the manager's wealth to changes in firm value as first proposed by Jensen and Murphy (1990). This approach is particularly convenient from an empirical point of view as it is based on firm value changes, which are easy to measure. In contrast, an output-based PPS measure must isolate that output process which is most directly attributable to the manager. If firm value is linear in output $X_t$, this simplification is inconsequential as value-based PPS would be equivalent to direct, output-based PPS, such as $\beta$. However, growth options can lead to a non-linear relationship between firm value and output. Thus there can be a wedge between output-based and value-based PPS in firms with growth opportunities .

In our model, as in He (2011), the manager's dollar value-based PPS is equal to the sensitivity of the manager's dollar continuation value to changes in firm value, $b(X)$. Under

[6] Given a performance measure $Y$, a standard way of measuring PPS is $\Delta$Manager's Wealth/$\Delta Y$. The continous time analog to this measure is $dV/dY$ since $V$ measures the dollar value of the manager's wealth. Since $dZ \cdot dt = 0$ and $dZ^2 = dt$, we have

$$\frac{dV}{dY} = \frac{dV}{dY}\frac{dZ}{dZ} = \frac{\frac{\sigma}{\mu} X k g'(a^*(X))}{\sigma_Y},$$

where the numerator is the volatility of $V$ given in Equation (12) and $\sigma_Y$ is the volatility of $Y$.

the optimal contract, this quantity is given by:

$$\phi^*(X) = \frac{\beta^*(X)}{b'(X)} = \frac{g'(a^*(X))k}{\mu b'(X)}. \tag{31}$$

with an equivalent formula for post-investment value-based PPS $\hat{\phi}^*(X)$. Note that while $\phi^*(X)$ is closely related to $\beta^*(X)$, it is scaled by the slope of the value function in output $b'(X)$. Thus, the presence of growth options affects $\phi^*(X)$ by changing both $\beta^*(X)$ and $b'(X)$. As we show in the next proposition, the wedge between $\beta^*$ and $\phi^*$ induced by $b'(X)$ can lead the two measures of PPS to respond in opposite ways to changes in the size of growth options. Specifically, the comparative static of $\beta^*(X)$ with respect to $\hat{k}$ can have the opposite sign as that for $\phi^*(X)$.

**Proposition 3.** *If the cost of effort is increasingly convex, $g'''(a) > 0$, and effort is interior, then*

$$Sign\left(\frac{\partial \beta^*(X)}{\partial \hat{k}}\right) \neq Sign\left(\frac{\partial \phi^*(X)}{\partial \hat{k}}\right). \tag{32}$$

To see the intuition for the result, suppose that optimal effort increases in the size of the investment opportunity. Since $\beta^*(X)$ is monotonic in $a^*(X)$, so that the increase in optimal effort corresponds to an increase in output-based PPS. In contrast, $\phi^*(X)$ can decrease in the size of the investment opportunity. This decrease occurs because $b'(X)$ increases in the size of investment opportunity meaning that the sensitivity of firm to output increases.[7] This sensitivity is not due to the manager's effort choice and therefore the manager does not have to be compensated (punished) as much for increases (decreases) in firm value. Thus, in firms with valuable growth options, it is optimal to set output-based PPS to a relatively low

[7]Note that $\phi^*(X)$ is a ratio of $\beta^*(X)$ and $b'(X)$. The condition that additional incentives are costly, i.e., $g(a)$ is increasingly convex, means that $\beta^*(X)$ increases less in $\hat{k}$ than $b'(X)$. This is because the optimal effort increases by less the more convex the effort cost.

level. In this case, a low $\phi^*(X)$ does not mean that the manager's incentives are low-powered but rather that a strong response of firm value to output allows the principal to set a low value-based PPS.

We emphasize here that although output- and value-based measures of performance can behave differently, either can be used to implement the optimal contract. Thus, our argument for the use of output-based PPS is distinct from that of Hölmstrom (1979) which points out that optimal contracts are always based on the most informative signal of an agents effort choice. In our setting, both performance measures are equally informative. However, output-based PPS gives a direct measure of incentives, whereas value-based PPS is composite of incentives and the effect of investment opportunities on firm value. We believe that this is related to a broader point: value-based PPS could be a biased measure of incentives whenever firm value is non-linear in output.

We now proceed to study the behavior of PPS at the moment of investment. When the firm exercises the growth option two changes occur that both affect PPS. First, an increase in capital increases both the cost and the benefit of effort. Second, a decrease in the remaining growth options decreases the benefit of effort. Thus, it is not immediately clear what will happen to PPS at investment. In the next proposition, we give conditions under which an increase or a decrease will occur.

**Proposition 4.** *The manager's power of incentives, measured by either $\beta$ or $\phi$, increases at investment if:*

$$g'(\hat{a}^*(\overline{X}))\hat{k} > g'(a^*(\overline{X}))k$$

*and decreases otherwise.*

Proposition 4 states that incentives increase at investment when the drop of the implemented effort at investment is sufficiently small relative to the inverse of the size of the growth option. This is the case for firms with low costs of effort (low agency conflicts) and large growth opportunities.

It is interesting to note that the condition in Proposition 4 is identical to the one given in Proposition 2 for the negative sign of the effect of risk aversion on the investment threshold. This means that the response of the manager's incentives to investment can be linked to the distortion in investment timing due to agency conflicts. Specifically, our model predicts that the power of incentives decreases at investment if moral hazard conflicts lower the investment threshold and increases at investment if moral hazard conflicts raises the investment threshold. Moreover, this relationship does not depend on whether or not the manager's incentives are measured by output- or value-based metrics.

### 4.3 Examples

We now return to the examples we discuss in the introduction to demonstrate how the implications of the previous two subsections could manifest in practice. It will be convenient to rearrange Inequality (27) to the following form

$$\frac{g'(\hat{a}^*(\overline{X}))}{g'(\hat{a}^*(\overline{X}))} \leq \frac{k}{\hat{k}}. \tag{33}$$

First, we consider a startup firm with a small amount of initial capital choosing the optimal time to substantially increase its capital stock and begin production. An example of this type of investment decision is when a growing private firm goes public in order to raise financing for a large increase in capital. The IPO literature has treated this decision as a real option, for example see Pástor and Veronesi (2005), Pástor, Taylor, and Veronesi

(2009), and Bustamante (2011). In this case, the capital stock after investment is much larger than before investment, $\hat{k} \gg k$, so that the right-hand side of Inequality (33) is close to zero. Note that the left-hand side of the inequality, the ratio of the manager's marginal cost of effort before and after investment, is always strictly positive. Thus the inequality is violated and Proposition 2 states that an increase in the severity of the moral hazard problem raises the investment threshold. Intuitively, if the startup firm is not subject to a moral hazard problem prior to investment, then a relatively large post-investment moral hazard problem will delay investment in production. Moreover, Proposition 4 indicates that while the manager's effort may decrease after the expansion, PPS increases.

Now we examine the example of one firm considering the acquisition of another firm. The literature on acquisitions has modeled this decision as a real option, see for example Lambrecht (2004), Morellec and Zhdanov (2005), and Hackbarth and Miao (2012). Often, the acquiring firm is much larger than the target firm. In such acquisitions, the capital stock after investment is not much larger than before investment, $\hat{k} - k \ll k$, so that the right-hand side of inequality (33) is close to one. The HJB equations together with the smooth-pasting condition imply that optimal effort always decreases at investment, so that the left-hand side of Inequality (33) is always strictly below one. Thus, the inequality is satisfied and an increase in the severity of the moral hazard problem lowers the threshold at which the acquisition takes place. The intuition here is that the acquisition allows the firm to grow its cash flows without requiring its manager to work more. This in turn allows the firm to save on incentive costs, so that when the severity of the moral hazard problem is more severe, the acquisition is optimally undertaken at a lower threshold in productivity. Furthermore, Proposition 4 implies that PPS decreases after the acquisition consistent with Harford and Li (2007) who find that CEO pay becomes "detached" from performance after an acquisition.

# 5 Other Investment Technologies

In this section, we enrich our model to consider other specifications of the investment problem. The main goal of this exercise is to determine under what conditions an increase in the severity of the moral hazard problem as measured by the manager's risk aversion, $\gamma$, leads to increases in investment. To that end, we make the following modifications to the model of the previous sections. First, productivity now follows a general diffusion of the form:

$$dX_t = a_t \mu(X_t)dt + \sigma(X_t)dZ_t,$$

where the drift and volatility terms, $\mu$ and $\sigma$, are continuously differentiable functions of productivity $X$. We maintain the restriction that effort must fall in the interval $a_t \in [0, 1]$; however, the cost of effort is now given by the general function $G(X_t, K_t, a_t)$ such that $G$ is twice continuously differentiable in it arguments and convex in effort $a_t$. Next, the firm's cash flows are given by a general function $\pi(X_t, K_t)$, which may exhibit increasing or decreasing returns to scale, and may depend on either the increment or the level of productivity as well. Finally, capital accumulates according to:

$$dK_t = (I_t - \delta K_t)dt,$$

where $\delta$ is capital depreciation and investment $I_t$ is at the cost $C(X_t, K_t, I_t)dt$ that may feature convex adjustment cost, partial reversibility, and stock fixed costs. A contract in this more general setting is then a triple $(a_t, c_t, I_t)$ consisting of a recommended effort level

$a_t$, a compensation plan $c_t$, and an investment rule $I_t$. In the following subsections we give a heuristic analysis of the generalized investment model, with formal proofs provided in the Appendix.

Note that the arguments leading to a characterization of the no-savings condition and incentive-compatibility conditions did not depend on a specification of the investment technology. Consequently, continuation utility arising from a contract without savings under this more general model must be a martingale and satisfy $u_c(c_t, a_t) = -\gamma r W_t$. The incentive compatibility condition is then:

$$a_t = \arg\max_{\tilde{a}} \left\{ u(c_t, \tilde{a}) + \beta_t(-\gamma r W_t)(\tilde{a} - a_t)\mu(X_t) \right\}, \tag{34}$$

which implies that:

$$\beta_t = \frac{1}{\mu(X_t)} G_a(X_t, K_t, a_t)$$

and

$$dW_t = \frac{\sigma(X_t)}{\mu(X_t)}(-\gamma r W_t) G_a(X_t, K_t, a_t) dZ_t.$$

The characterization of incentive compatibility given above allows us to proceed to analyze the effect moral hazard on investment in this more general setting.

For many of the models subsumed by our general setup, the optimal investment policy is an increasing function of the investor's marginal value of capital, commonly referred to as Tobin's $Q$. For this class of models, including the neoclassical and capacity choice models, the effect of the agency problem on optimal investment operates entirely through $q$. Thus, to determine the effect of the moral hazard problem on optimal investment in these models, it is sufficient to determine its effect on the marginal value of capital. We now show that increasing the severity of the moral hazard problem decreases the marginal value of capital and hence curtails investment.

To determine the effect of the moral hazard problem on the investor's marginal value of capital, we again apply the method of comparative statics developed in DeMarzo and Sannikov (2006). Since the dynamics of continuation utility remain essentially unchanged from the previous sections, the investor's value function is still additively separable as $B(X, K, V) = b(X, K) - V$. Taking as given the optimal investment and effort policies $I^*$ and $a^*$, an application of Ito's formula, the envelope theorem, and the Feynman-Kac formula given in Lemma 4 of the Appendix yields the following expression for the derivative of the marginal value of capital, $b_k$, with respect to the manager's risk aversion $\gamma$:

$$b_{K\gamma} = E\left[\int_0^\infty e^{-(r+\delta)} h_{K\gamma}(X_t, K_t, a^*(X_t, K_t))dt \big| X_0, K_0\right], \tag{35}$$

where $h(X, K, a)$ is defined as in Equation (18) and represents the total cash flow to the firm net of effort and incentive costs. Equation (35) states that the derivative of the marginal value of capital with respect to $\gamma$ is just the expected present value of all future derivatives of the marginal products of capital with respect to $\gamma$. For any given point $(X, K, a)$, it is straightforward to compute the derivative of the marginal product of capital with respect to $\gamma$ to find:

$$h_{K\gamma}(X, K, a) = -r\left(\frac{\sigma(X)}{\mu(X)}\right)^2 G_a(X, K, a)G_{aK}(X, K, a) \leq 0. \tag{36}$$

Substituting Inequality (36) into Equation (35) implies the following proposition.

**Proposition 5.** *The investor's marginal value of capital $b_K$ is decreasing in the manager's risk aversion $\gamma$.*

Proposition 5 confirms the intuition of the previous literature (e.g., DeMarzo et al. (2012) and DeMarzo and Fishman (2007)) that the moral hazard problem decreases the marginal value capital. This in turn implies that if the optimal investment policy can be expressed as an increasing function of the marginal value of capital otherwise independent of the severity of the moral hazard problem, then optimal investment decreases the severity of the moral hazard problem. For example, in the neoclassical model with convex adjustment costs ($C_{II}(X, K, I) > 0$), the optimal investment rate equates the marginal value of capital with the marginal cost of capital:

$$b_K(X, K, a^*) = C_I(X, K, I^*). \tag{37}$$

Since investment costs are independent of the manager's risk aversion, we can differentiate both sides of Equation (37) to find:

$$I^*_\gamma = \frac{b_{K\gamma}(X, K, a^*)}{C_{II}(X, K, I^*)} \leq 0,$$

so that investment decreases with the severity of the moral hazard problem. Similarly, in a capacity choice model with partial reversibility, as in Abel and Eberly (1996), the optimal investment policy is to invest (divest) only when the marginal value of capital is greater (less) than the marginal purchase price of capital. For any given level of productivity $X$, increasing the severity of the moral hazard problem through an increase in $\gamma$ decreases the marginal value of capital. Thus both the thresholds in productivity at which the firm invests and divests increase with the severity of the agency problem.

Although the $Q$-theory and capacity choice models are standard ways to describe aggregate investment patterns, firm-level investment is often lumpy. Moreover, firms often have a limited quantity of investment opportunities. Our setting captures these two features by modeling investment as a single real option (a finite number of real options would give qualitatively similar results). This leads to important differences with the literature in the way in which investment is determined. First, in our model, the marginal value of capital is not a sufficient statistic for investment, rather investment occurs when the average value of new capital reaches an upper threshold. Second, the firm has a limited quantity of investment opportunities. This means that marginal value of capital changes at the moment of investment so that the average value of new capital is not proportional to the marginal value of capital. As a result, although moral hazard decreases the marginal value of capital, it can increase the average value of new capital and thus cause over-investment.

# 6 Conclusion

We present a model of real options and dynamic moral hazard. We find that the effect of agency conflicts on investment timing depends on the severity of the conflict. When the moral hazard problem is less severe, the optimal contract will implement high effort, but raise the investment threshold. When the moral hazard problem is more severe, the optimal contract will implement lower effort but will call for a lower investment threshold. The finding that moral hazard may increase investment is new and provides an alternative to empire-building or managerial hubris-based explanations of over-investment. Although the primary real options model we consider is fairly simple, we show that the main intuition carries over into more realistic settings, so long as the optimal investment path is lumpy.

Our model also admits results on pay-performance sensitivity. Like investment, the effect of moral hazard on pay-performance sensitivity depends on the severity of the moral hazard problem. When the moral hazard problem is less severe, pay-performance sensitivity increases after investment. When the moral hazard problem is more severe, pay-performance sensitivity decreases with investment. These results link pay-performance sensitivity, which is easily measurable, with the nature the distortion on investment timing imposed by moral hazard, which is more difficult to measure.

Our results provide guidance to future empirical work on pay-performance sensitivity. We show that in the presence of growth options, there is a wedge between output-based and value-based measures of incentives. In fact, if the manager's cost of effort is increasingly convex, value-based pay-performance sensitivity may decrease even though true incentives (i.e., output-based pay-performance sensitivity) increase. Thus, it is important to control for the presence of growth options when using value-based measures of pay-performance sensitivity as a proxy for the level of incentives. One could take a reduced-form approach to evaluate this relation. In addition, our model provides a tractable framework for structural estimation of the quantitative importance of moral hazard distortions to investment.

Our general model could be extended and applied to specific contexts following the real option literature without agency conflicts, e.g., mergers and acquisitions, real estate development, IPOs, and venture-capital financing. In each case, there are important institutional details that we have omitted from the model for the sake of clarity. However, these details may provide interesting new results and implications. For example, in mergers and acquisitions, the investment may depend on the productivity of both the bidding and target firms. An important feature of real estate development that may interact with the agency conflict we consider is that investment typically requires time-to-build. Finally, in IPOs venture-capital financing, the manager may have some private information that affects the value growth option.

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# Appendix

## A Proofs

*Proof of Lemma 1.* Consider an arbitrary contract $\Pi = (\{c_t, a_t\}, \tau)$ and suppose the solution to the manager's optimization problem (1) for this contract is given by $\{\tilde{c}_t, \tilde{a}_t\}$ and the manager's associated value for this contract is $\tilde{W}_0$.

Now consider the alternative contract $\tilde{\Pi} = (\{\tilde{c}_t, \tilde{a}_t\}, \tau)$. Note that under this contract the manager again gets utility $\tilde{W}_0$ from the consumption effort pair $\{\tilde{c}_t, \tilde{a}_t\}$. We claim that the solution to manager's optimization problem (1) is again $\{\tilde{c}_t, \tilde{a}_t\}$. Indeed suppose it is not and that there is a alternative feasible consumption effort pair $\{\check{c}_t, \check{a}_t\}$ such that this policy yields utility $\check{W}_0 > \tilde{W}_0$ to the manager. The consumption effort pair $\{\check{c}_t, \check{a}_t\}$ is also feasible under the original contract $\Pi$ since:

$$\begin{aligned}
\lim_{t\to\infty} E\left[e^{-rt}\int_0^t (c_s - \check{c}_s)ds\right] &= \lim_{t\to\infty}\left(E\left[e^{-rt}\int_0^t (c_t - \tilde{c}_t)dt\right] + E\left[e^{-rt}\int_0^t (\tilde{c}_s - \check{c}_s)ds\right]\right) \\
&= \lim_{t\to\infty} E\left[e^{-rt}\int_0^t (c_s - \tilde{c}_s)ds\right] + \lim_{t\to\infty} E\left[e^{-rt}\int_0^s (\tilde{c}_s - \check{c}_s)ds\right] \\
&= 0.
\end{aligned}$$

Thus, the manager could achieve utility $\check{W}_t > \tilde{W}_t$ under the original contract $\Pi$, a contradiction.

Finally note that the investor is achieves the same value under the new contract $\tilde{\Pi}$ as under the original contract $\Pi$, since effort and investment are unchanged, and the traversality condition implies that the two consumption streams have the same present value.

□

*Proof of Lemma 2.* Suppose $S_t = \mathcal{S}$ and recall the definition of $W_t(\Pi, \{X_s, K_s\}_{s\le t}; \mathcal{S})$ and let $\{\tilde{c}, \tilde{a}\}$ solve problem (3). We claim that $\{\tilde{c} - rS, \tilde{a}\}$ solves problem (3) for $S_t = 0$. This plan gives the manager a utility of $W_t(\Pi, \{X_s, K_s\}_{s\le t}; 0) = e^{\gamma r \mathcal{S}} W_t(\Pi, \{X_s, K_s\}_{s\le t}; \mathcal{S})$. Suppose there is some alternative $\{\check{c}, \check{a}\}$ that yields a higher utility to the agent $\check{W}_t(\Pi, \{X_s, K_s\}_{s\le t}; 0) > W_t(\Pi, \{X_s, K_s\}_{s\le t}; 0)$. Now consider the plan $\{\check{c} + rS, \check{a}\}$ and note that this plan is feasible under $S_t = \mathcal{S}$ but under this plan the manager can achieve the following utility:

$$
\begin{aligned}
\check{W}_t(\Pi, \{X_s, K_s\}_{s\le t}; 0) &= e^{-\gamma r \mathcal{S}} \check{W}_t(\Pi, \{X_s, K_s\}_{s\le t}; 0) \\
&\ge e^{-\gamma r \mathcal{S}} W_t(\Pi, \{X_s, K_s\}_{s\le t}; 0) \\
&= W_t(\Pi, \{X_s, K_s\}_{s\le t}; 0),
\end{aligned}
$$

which contradicts the optimality of the $\{\tilde{c}, \tilde{a}\}$. □

*Proof of Proposition 1.* We show that the candidate policies are indeed optimal for the investor. Note that the compensation policy $c_t$ is pinned down by the no-savings condition. Define the stopped gain process by:

$$
G_t = \int_0^t e^{-rs}(X_t K_t - c_t)dt + e^{-rt} B(X_t, V_t) + \mathbb{I}(t \ge \tau)(e^{-rt}(\hat{B}(X_t, V_t) - B(X_t, V_t)) - e^{-r\tau} p).
$$

When $V_t$ evolves according to (12), Ito's formula gives the following dynamics:

$$
\begin{aligned}
e^{rt} dG_t = \Bigg[ & X_t K_t - \frac{1}{2}\theta X_t K_t a^2 + -rV_t + a_t \mu X_t \mathcal{B}_X + \frac{1}{2}\sigma^2 X_t^2 \mathcal{B}_{XX} \\
& + \frac{1}{2}\gamma r \left(\frac{\sigma}{\mu} X_t \hat{k} g'(a)\right)^2 \mathcal{B}_V + \frac{\sigma^2}{\mu} X^2 \hat{k} g'(a) \mathcal{B}_{XV} + \frac{1}{2}\left(\frac{\sigma}{\mu} X \hat{k} g'(a)\right)^2 \mathcal{B}_{VV} - r\mathcal{B} \Bigg] dt \\
& + \left(\mathcal{B}_X + \frac{\sigma}{\mu} X_t K_t g'(a_t) \mathcal{B}_V\right) \sigma X_t dZt + \mathbb{I}(t = \tau)(\hat{B}(X_t, V_t) - B(X_t, V_t) - p),
\end{aligned}
$$

where $\mathcal{B}(X_t, V_t) = B(X_t, V_t) + \mathbb{I}(t \geq \tau)(\hat{B}(X_t, V_t) - B(X_t, V_t))$. Note that the drift term (the term in the square brackets) is clearly negative for any alternative policy, while it is zero for the candidate policy. Now examine the last term. Under the candidate policy, this term is zero due to the value-matching condition. Under any alternative investment time $\tau$, this term is negative due to the smooth-pasting condition and the concavity of $\hat{B}(X_t, V_t) - B(X_t, V_t)$. As a result, the process $G_t$ is a martingale under the proposed contract and a supermartingale otherwise. The rest of the argument proceeds along the standard lines. □

*Proof of Proposition 2.* The following lemmas aid in the proof of the proposition. For ease of exposition, we leave their proofs to the end of the main argument.

**Lemma 4.** *Suppose $X_t$ evolves according to $dX_t = \mu(X_t)dt + \sigma(X_t)dZ_t$. Then for some bounded functions $f\colon (0, Y] \to \mathbb{R}$, $r\colon (0, Y] \to \mathbb{R}^+$, and $\Omega\colon \mathbb{R} \to \mathbb{R}$, a function $F : (0, Y] \to \mathbb{R}$ solves both:*

$$r(X)F(X) = f(X) + \mu(X)F_X(X) + \frac{1}{2}\sigma(X)^2 F_{XX}(X), \tag{A.1}$$

*with a boundary condition $F(Y) = \Omega(Y)$ and*

$$F(X) = E\left[\int_0^\tau e^{-\int_0^t r(X_s)ds} f(X_t)dt + e^{-\int_0^\tau r(X_s)ds}\Omega(Y)\big|X_0 = X\right], \tag{A.2}$$

*where $\tau = \inf\{t|X_t \geq Y\}$.*

**Lemma 5.** $\hat{b}_{XX}(\overline{X}) - b_{XX}(\overline{X}) \neq 0$.

**Lemma 6.** *There exists $\epsilon > 0$ such that $\hat{b}_X(X) - b_X(X) > 0$ for all $X \in (\overline{X} - \epsilon, \overline{X})$.*

**Lemma 7.** *There exists $\epsilon > 0$ such that when $X \in (\overline{X} - \epsilon, \overline{X})$ we have sign$(\hat{b}_\gamma(X) - b_\gamma(X)) =$ sign$\left(kg'(a^*(\overline{X}) - \hat{k}g'(\hat{a}^*(\overline{X}))\right)$.*

We now proceed to the main argument. The first step in determining the sign of $\overline{X}_\gamma$ is to differentiate the smooth-pasting condition with respect to $\gamma$ to get:

$$\overline{X}_\gamma(\hat{b}_{XX}(\overline{X}) - b_{XX}(\overline{X})) = \hat{b}_{\gamma X}(\overline{X}) - b_{\gamma X}(\overline{X}). \tag{A.3}$$

Lemma 6 together with an application of the one-sided version of l'Hopital's rule yields the following expression for $\overline{X}_\gamma$:

$$\overline{X}_\gamma = -\frac{\hat{b}_{X\gamma}(\overline{X}) - b_{X\gamma}(\overline{X})}{\hat{b}_{XX}(\overline{X}) - b_{XX}(\overline{X})} = \lim_{X \uparrow \overline{X}} -\frac{\hat{b}_{X\gamma}(X) - b_{X\gamma}(X)}{\hat{b}_{XX}(X) - b_{XX}(X)} = \lim_{X \uparrow \overline{X}} -\frac{\hat{b}_{\gamma}(X) - b_{\gamma}(X)}{\hat{b}_{X}(X) - b_{X}(X)}, \tag{A.4}$$

so that determining the sign of $\overline{X}_\gamma$ is equivalent to determining the sign of the last limit above.

If $\frac{g'(\hat{a}^*(\overline{X}))}{g'(a^*(\overline{X}))} < \frac{k}{\hat{k}}$, Lemmas 6 and 7 imply there exists $\epsilon > 0$ such that:

$$-\frac{\hat{b}_{\gamma}(X) - b_{\gamma}(X)}{\hat{b}_{X}(X) - b_{X}(X)} < 0$$

for all $X \in (\overline{X} - \epsilon, \overline{X})$, which in turn implies:

$$\lim_{X \uparrow \overline{X}} -\frac{\hat{b}_{\gamma}(X) - b_{\gamma}(X)}{\hat{b}_{X}(X) - b_{X}(X)} \geq 0,$$

since $\hat{b}_\gamma(X) - b_\gamma(X)$ and $\hat{b}_X(X) - b_X(X)$ are nonzero and continuous. Thus, $\overline{X}_\gamma \leq 0$. If $\frac{g'(\hat{a}^*(\overline{X}))}{g'(a^*(\overline{X}))} < \frac{k}{\hat{k}}$, a similar argument shows $\overline{X}_\gamma \geq 0$.

*Proof of Lemma 4.* The proof essentially follows the proof of DeMarzo and Sannikov (2006) for Lemma 4. Suppose $V$ solves Equation (A.1) and define the process $H_t$ by:

$$H_t = \int_0^t e^{-\int_0^s r(X_u)du} f(X_s)ds + e^{-\int_0^t r(X_s)ds} V(X_s).$$

An application of Ito's formula gives the dynamics for $H_t$:

$$e^{\int_0^t r(X_s)ds} dH_t = (f(X_t)+\mu(X_t)V_X(X_t)+\frac{1}{2}\sigma(X_t)^2 V_{XX}(X_t)-r(X_t)V(X_t))dt+\sigma(X_t)V(X_t)dZ_t.$$

By Equation (A.1), the drift in the above dynamics is zero, so that $H_t$ is a martingale. Since $V(X)$ is bounded on $[0, \overline{X}]$, $H_\tau$ is a martingale and:

$$\begin{aligned} V(X_0) = H_0 = E[H_\tau | X_0] &= E\left[\int_0^\tau e^{-\int_0^t r(X_s)ds} f(X_t)dt + e^{-\int_0^\tau r(X_s)ds} V(X_\tau) \middle| X_0\right] \\ &= E\left[\int_0^\tau e^{-\int_0^t r(X_s)ds} f(X_t)dt + e^{-\int_0^\tau r(X_s)ds} W(Y) \middle| X_0\right]. \end{aligned}$$

where the last equality follows from the definition of the stopping time $\tau$ and the boundary condition $V(Y) = W(Y)$. □

*Proof of Lemma 5.* Assume that $\hat{b}_{XX}(\overline{X}) - b_{XX}(\overline{X}) = 0$. We show that this leads to a contradiction. First, differentiate the smooth-pasting condition (23) with respect to $p$ to obtain:

$$b_{Xp}(\overline{X}) = -(\hat{b}_{XX}(\overline{X}) - b_{XX}(\overline{X}))\overline{X}_p.$$

$\hat{b}_{XX}(\overline{X}) - b_{XX}(\overline{X}) = 0$ implies that $b_{Xp}(\overline{X}) = 0$. Next, differentiate the ODE (21) with respect to $X$ and $p$ to get an ODE for $b_{Xp}$:

$$(r - a^*(X)\mu)b_{Xp} = (a^*(X)\mu + \sigma^2)Xb_{XXp} + \frac{1}{2}\sigma^2 X^2 b_{XXXp}.$$

Lemma 4 then implies:

$$b_{Xp}(X) = E\left[e^{-\int_0^\tau (r-a^*(\tilde{X}_s)\mu)ds} b_{Xp}(\overline{X})|\tilde{X}_0 = X\right] = 0,$$

where $\tilde{X}$ is a process with the following dynamics:

$$d\tilde{X} = (a^*(\tilde{X})\mu + \sigma^2)\tilde{X}dt + \sigma\tilde{X}dZ.$$

Since $b_{Xp}(X) = 0$, then $b_p(X)$ must be a constant in $X$. By differentiating the value-matching condition (22) with respect to $p$, we get:

$$b_p(\overline{X}) = (\hat{b}_X(\overline{X}) - b_X(\overline{X}))\overline{X}_p - 1 = -1,$$

where the second equality follows from the smooth-pasting condition (23). Differentiating the ODE (21) with respect to $p$ to gives an ODE for $b_p$:

$$rb_p(X) = a(X)\mu X b_{Xp}(X) + \frac{1}{2}\sigma^2 X^2 b_{XXp}(X).$$

Lemma 4 then implies:

$$b_p(X) = E\left[e^{-r\tau} b_p(\overline{X})|X_0 = X\right].$$

In particular,

$$\lim_{X\to 0} b_p(X) = \lim_{X\to 0} E\left[e^{-r\tau} b_p(\overline{X})|X_0 = X\right] = 0,$$

as $X = 0$ is an absorbing point for the process $X_t$. This means that $b_p(X)$ cannot be a constant, which is a contradiction. $\square$

*Proof of Lemma 6.* Suppose there does not exist $\epsilon > 0$ such that $b_X(X) < \hat{b}_X(X)$ for all $X \in (\overline{X} - \epsilon, \overline{X})$, then for all $\epsilon > 0$ there exists $X \in (\overline{X} - \epsilon, \overline{X})$ such that $b_X(X) \geq \hat{b}_X(X)$. Now since $b_X(\overline{X}) = \hat{b}_X(\overline{X})$ and $b_X$ and $\hat{b}_X$ are continuous, this implies that there exists $\epsilon > 0$ such that $b_X(X) \geq \hat{b}_X(X)$ for all $X \in (\overline{X} - \epsilon, \overline{X})$. This implies that for $X \in (\overline{X} - \epsilon, \overline{X})$, we have:

$$\hat{b}(X) - b(X) = \hat{b}(\overline{X}) - b(\overline{X}) - \int_X^{\overline{X}} (\hat{b}_X(x) - b_X(x))dx \geq p,$$

which is a contradiction to the definition of $\overline{X}$. $\square$

*Proof of Lemma 7.* First, we differentiate the HJB equations (17) and (21) with respect to $\gamma$ to get:

$$r\hat{b}_\gamma = \hat{h}_\gamma(X, \hat{a}^*(X)) + \hat{a}^*(X)\mu X \hat{b}_{X\gamma} + \frac{1}{2}\sigma^2 X^2 \hat{b}_{XX\gamma},$$

and:

$$rb_\gamma = h_\gamma(X, a^*(X)) + a^*(X)\mu X b_{X\gamma} + \frac{1}{2}\sigma^2 X^2 b_{XX\gamma},$$

subject to $b_\gamma(\overline{X}) = \hat{b}_\gamma(\overline{X})$. For the remainder of the proof of this lemma, we will suppress the dependence of $h_\gamma$ and $\hat{h}_\gamma$ on $a^*$ and $\hat{a}^*$ to ease notation. Note that $\hat{h}_\gamma(\overline{X}) \geq \hat{h}_\gamma(\overline{X})$ is equivalent to $\frac{g'(\hat{a}^*(\overline{X}))}{g'(a^*(\overline{X}))} < \frac{k}{\hat{k}}$.

Let $X^*$ and $\hat{X}^*$ be given by the following dynamics:

$$dX^* = a^*(X^*)\mu X^* dt + \sigma X^* dZ$$
$$d\hat{X}^* = \hat{a}^*(\hat{X}^*)\mu \hat{X}^* dt + \sigma \hat{X}^* dZ.$$

Applying Lemma 4 we have:

$$b_\gamma(X) = E\left[\int_0^\tau e^{-rt} h_\gamma(X_t^*)dt + e^{-r\tau}\hat{b}_\gamma(\overline{X})\big|X_0^* = X\right]$$

and

$$\hat{b}_\gamma(X) = E\left[\int_0^{\hat{\tau}} e^{-rt} \hat{h}_\gamma(\hat{X}_t^*)dt + e^{-r\hat{\tau}}\hat{b}_\gamma(\overline{X})\big|\hat{X}_0^* = X\right],$$

where $\tau = \inf\{t|X_t^* \geq \overline{X}\}$ and $\hat{\tau} = \inf\{t|\hat{X}_t^* \geq \overline{X}\}$. Subtracting we get:

$$\hat{b}_\gamma(X) - b_\gamma(X) = E\left[\int_0^{\hat{\tau}} e^{-rt} \hat{h}_\gamma(\hat{X}_t^*)dt - \int_0^\tau e^{-rt} h_\gamma(X_t^*)dt + (e^{-r\hat{\tau}} - e^{-r\tau})\hat{b}_\gamma(\overline{X})\big|X_0^* = \hat{X}_0^* = X\right].$$

Now by continuity and the fact that $\hat{\tau}, \tau \overset{\text{a.s.}}{\to} 0$ as $\hat{X}_0^*, X_0^*, \to \overline{X}$, there exists $\epsilon > 0$ such that when $\hat{X}_0^*, X_0^* \in (\overline{X} - \epsilon, \overline{X})$ we have:

$$\operatorname{sign}(\hat{b}_\gamma(X) - b_\gamma(X)) = \operatorname{sign}(\hat{h}_\gamma(\overline{X}) - h_\gamma(\overline{X})) \tag{A.5}$$

$$= \operatorname{sign}\left(kg'(a^*(\overline{X}) - \hat{k}g'(\hat{a}^*(\overline{X}))\right), \tag{A.6}$$

which is the desired result.

□

□

*Proof of Proposition 3.* To consider the effect of the size of investment opportunities on incentives, we analyze the effect of increasing post-investment capital $\hat{k}$.

First, consider the effect on pre-investment $\beta_t$:

$$\frac{d\beta_t}{d\hat{k}} = \frac{1}{\mu}kg''(a^*(X_t))\frac{da^*(X_t)}{d\hat{k}}.$$

Consider next the effect of $\hat{k}$ on $\phi_t$:

$$\frac{d\phi_t}{d\hat{k}} = -\frac{\mu^2}{(\mu^2 + \gamma r \sigma^2 X_t k g''(a^*(X_t)))^2} \gamma r \sigma^2 X_t k g'''(a^*(X_t)) \frac{da^*(X_t)}{d\hat{k}}.$$

So:

$$\text{sign}\left(\frac{d\phi_t}{d\hat{k}}\right) = \text{sign}\left(-g'''(a^*(X_t))\frac{da^*(X_t)}{d\hat{k}}\right).$$

This shows that the sign of the effect of $\hat{k}$ on $\phi_t$ is the same as on $\beta_t$ if $g'''(a^*) < 0$ and the opposite if $g'''(a^*) > 0$. $\square$

*Proof of Proposition 4.* The result for $\beta$ follows directly by comparing pre- and post-investment $\beta_t$ at the moment of investment. The relation for $\phi$ is found similarly using the smooth-pasting condition (23).

$\square$