A Pricing Model for Clearing End of Season Retail Inventory

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ABSTRACT

Setting prices for clearing retail inventories of fashion goods is a difficult task with the potential for costly mistakes. The problem is exacerbated by a declining demand pattern that is less sensitive to markdowns towards the end of the selling season. In this article, we present new discrete-time models for setting clearance prices with time-dependent price sensitivity of demand. When demand is deterministic, we develop a procedure for computing optimal prices and show that decreasing price-sensitivity leads to declining prices. We also develop models with stochastic demand in which demand in different periods can be arbitrarily distributed, and obtain bounds on the optimal expected revenue and on optimal prices. A heuristic solution is proposed and its accuracy tested. We find that the penalty for choosing markdown price once and keeping it unchanged for the remainder of the selling season (called the single-discount strategy) is highest when price sensitivity declines moderately in later periods. Thus, retailers are justified in using a single-discount strategy both when price-sensitivity does not change appreciably over time and when it drops sharply in later periods. In virtually all cases, period-specific clearance prices obtained after solving an equivalent deterministic problem lead to higher expected revenue than the optimal expected revenue of a single discount strategy. We show that efforts to affect the joint distribution of demand are guaranteed to be beneficial when they result in convexly decreasing partial sums of demand, irrespective of the nature of demand correlations across different periods.

Subject Areas: Retail inventory management, inventory theory, pricing.

1 Introduction

With the rapid proliferation of retail SKUs (stock-keeping units), mitigating market mediation costs in the retail of fashion goods has become increasingly important for profitability of retailers. When facing less-than-expected sales, retailers try to recover as much revenue as possible via price markdowns. Mistakes can be very costly. For example, one large web retailer (Amazon.com) reported a $39 million charge in 1999 for unsold Christmas inventory (Vogelstein 2000). OfficeMax reported a "special markdown charge" of approximately $50 million to liquidate inventory in order to accelerate profitability (OfficeMax 1999). The Chief Executive Office of Ann Taylor Stores reported that 1996 markdown units were more than 40% of the inventory (Ann Taylor Stores Press Release 1997). Smith and Achabal (1998) report that the markdown dollars (the difference between the regular price and the actual sales dollars) is "often several hundred million dollars for major retailers."

A retailer must address two types of questions when faced with excess inventory near the end of the regular selling season. The first has to do with the timing of the start of the clearance period. That is, when should the retailer declare its inventory to be surplus and begin to markdown prices. The second decision relates to how much discount should be offered. The end of the clearance period is defined by an outdate set by management (Smith & Achabal 1998). This is when any remaining inventory is sold at salvage and new items arrive to replace the old ones on the store shelves. The problem is exacerbated by the fact that there exist multiple opportunities for markdowns and that both the magnitude and the price-sensitivity of demand decline as the selling season progresses. For large retail chains that have multiple outlets, clearance prices can be different at each store, based on local inventory and forecasted demand during the clearance period. This practice supports the observation that competitive response to price markdown is not significant (Zabel 1970). Therefore, models used for setting clearance prices may be constructed without accounting for competition.

There is substantial literature on the problem of setting prices for perishable inventories. Smith and Achabal (1998) divide this literature in three main categories. The first category is a body of work from the marketing literature in which authors have attempted to characterize how demand responds to price. Previous empirical research supports a multiplicative model with exponential price sensitivity (see Smith and Achabal 1998 for details), which is the model we use in this article. The second body of work deals with the issue of setting prices for clearance items as a function of the time remaining until the end of selling season and the remaining stock on hand. In this case, both the inventory at the start of the clearance season and the remaining length of the selling season are assumed to be determined exogenously. Concomitantly, the cost of acquiring inventory is treated as sunk. Articles by Smith and Achabal (1998) and Gallego and Van Ryzin (1994) contain good summaries of previous work along these lines. The third related research is the body of work dealing with the joint optimization of price and inventory decisions for seasonally demanded items. Recently, Petruzzi and Dada (1999) have reviewed a long line of articles dealing with this class of problems in a single period (newsvendor) setting. Note that there are close parallels between clearance pricing of retail inventories and pricing of perishable inventory of seats on an airplane or rooms in a hotel. The latter problems belong to the class of perishable asset revenue management (PARM) problems that have been the focus of considerable research activity in recent years (McGill and Van Ryzin 1999).

Smith and Achabal (1998) develop a deterministic demand model for pricing clearance items. They treat demand as a function of price, time, and inventory level. Lower inventories indicate smaller assortment, which can retard demand if it falls below a minimum inventory level that they call the “fixture fill” quantity. The model allows prices to be different at each store, based on local conditions such as inventory levels and demand rates.

Gallego and Van Ryzin (1994) assume Poisson distributed demand and allow the intensity of demand to depend on price. When the demand intensity is an exponential function of price, they obtain the price trajectory as an explicit function of time. They also show that the optimal prices obtained from the deterministic model are asymptotically optimal for the stochastic model. The optimal price in the deterministic model is a fixed price for the entire selling season.

Our approach has several key differences from previous research. The most significant new feature of our models is the inclusion of time-dependent price-sensitivity. The underlying logic is that customers are unlikely to buy fashion items even at deep discounts towards the end of the selling season (e.g., a customer only wants one swimming suit, and in cold climates, customers rarely buy swimming suits in winter). Our results are also quite different from earlier papers. For example, in contrast to Gallego and Van Ryzin’s study, in the deterministic version of our model, a fixed price strategy is not optimal. In fact, when price sensitivity declines in period index, so do the optimal prices.

The second major difference is that we allow the stochastic component of demand to have any arbitrary distribution in a multiplicative model. This contrasts with Gallego and Van Ryzin's Poisson distributed demand model. In fact, nearly all previous studies dealing with multiple opportunities to price perishable inventories have assumed a Markov demand model. Some recent examples of such articles are Feng and Gallego (1995), Zhao and Zheng (2000), and Feng and Xiao (2000).

Other differences are as follows. As compared to the Smith and Achabal (1998) study, the demand in our model does not depend on the amount of inventory on hand. However, demand does decline with time, both in overall size and in terms of sensitivity to price. In view of the difficulty of implementing a continuously changing price trajectory, we model time as a discrete variable, that is, our model has multiple, but finite, opportunities to set prices. This is consistent with industry practice. Large retailers typically set markdown prices at no more than two or three discrete points in time. Gallego and Van Ryzin (1994) and Smith and Achabal (1998), on the other hand, obtain price trajectories with prices varied continously.

In all of the models reported in this article, we are not concerned with how much inventory should be ordered at the start of the selling season. Our goal is to determine optimal prices. The multiperiod model can be used to find the start of the markdown portion of the selling season by considering the entire selling season. The period in which the first markdown occurs is then the start of the clearance season. Thus, the models and analysis presented here can be used to study a general class of problems involving pricing decisions for finite, non-replenishable inventory when price-sensitivity of demand is time-dependent.

This paper is organized as follows. Section 2 develops the deterministic and stochastic demand clearance pricing models assuming a single markdown opportunity. Section 3 explores models with multiple opportunities for markdown. Section 4 presents numerical examples and managerial insights. In section 5, we show that promotional and market data gathering activities that reduce the variability of the joint distribution of demand are beneficial, irrespective of the nature of demand correlation in different periods. Contributions of this paper are summarized in section 6.

2 Single Markdown Opportunity

We begin the analysis with a one-period model, i.e., the pricing decision is made only once at the start of the clearance period, and once selected, the price is not changed. The formulation of the one-period problem provides a foundation for the more complicated and realistic model with multiple markdowns. The analysis is presented in two parts. The first part deals with deterministic demand and the second with stochastic demand. We use the following common notation for both models and throughout this article.

I	Inventory at the start of the clearance season.
p	Discount price (decision variable).
s	Salvage value for leftover stock at the end of clearance season. Salvage s is net of any holding and disposal charges.
$y(p)$	$= Ke^{-\beta p} =$ Price-Demand model, where K is the scale factor and $\beta > 0$ is the price-sensitivity parameter.
$D(p)$	$= y(p)\xi =$ Total demand during clearance season as a function of price.
ξ	Random component of demand function ($\xi \geq 0$).
$\mu, \sigma^2, f(\cdot), F(\cdot)$	Mean, variance, probability density, and cumulative distribution function of ξ.
$d(p)$	$= \mu y(p) =$ Total demand during clearance season in the deterministic model as a function of price p.
$z(p)$	$= I/y(p) =$ A value of ξ, as a function of price p, for which the total clearance season demand equals I. For a fixed I, z is completely determined by p. The corresponding p is also known as the stock-clearing price.

Quantities of the item under consideration are continuous. Notice that demand decreases in price, and in the price-sensitivity parameter β (where $\beta > 0$), and increases in the scale factor K. [In this article, the terms increasing (decreasing) are used to indicate non-decreasing (non-increasing) relationships. Strictly increasing (decreasing) relationships are identified with the qualifier strict.] The parameter K can be interpreted as a measure of the size of the market – the amount that will be sold if price is set equal to zero. For the one-period problem, K and β are assumed to be constant over the entire clearance season. It is implicitly assumed that irrespective of the amount of leftover stock, all excess inventory can be salvaged at the end of the clearance season at $s > 0$ per unit. This is quite realistic because retailers have the option of either selling their leftover inventory to discounters, or donating to charity. The latter generates a positive cash flow through a tax writeoff.

In addition to the notation defined above, we use $\pi(\cdot)$ to denote the total revenue function. Superscripts "d" and "s" are used to denote the deterministic and the stochastic demand scenarios respectively. In our models, the inventory at the start of the clearance season is not a decision variable, hence π is shown as a function only of p. Readers should also note that Petruzzi and Dada (1999) include a similar analysis of the one-period problem when the price-demand model is represented by a power function, i.e., using our notation $y(p) = ap^{-b}$, with $a, b > 0$.

2.1 Single Markdown Opportunity With Deterministic Demand

In this deterministic or *riskless* environment, if the retailer picks a price p, it knows with certainty that the demand will be $d(p) = \mu y(p)$. Clearly, the retailer should never pick a price such that $d(p) > I$. This is because in every such situation it can extract extra revenue by charging a higher price for which $d(p) = I$. Therefore, we wish to solve the following optimization problem to find the optimal price p that maximizes total revenue.

$$\text{Maximize } \pi^d(p) = pd(p) + s(I - d(p)) \tag{1}$$

Subject to:

$$p : d(p) \leq I \tag{2}$$

The objective function does not contain a cost term signifying the purchase cost to the retailer. This is consistent with industry practice and underscores the fact that the original purchase cost of leftover items is a sunk cost. Constraint 2 simplifies to yield $p : p \geq -\ln(I/\mu K)/\beta$. Since p must also be at least s, the constraint is meaningful only if $-\ln(I/\mu K)/\beta \geq s$, i.e., when there exists a price level $p > s$ at which the retailer could sell its entire surplus inventory I.

Setting the first derivative of $\pi^d(p)$ to zero, we find that it has two roots: either $p = 1/\beta + s$ or $p = \infty$. It is easy to verify that the second derivative of π^d is negative only when $p < 2/\beta + s$ which means that only $p = 1/\beta + s$ is a local maximum. Therefore, the optimum price is the unconstrained solution, $p = 1/\beta + s$ if $\mu y(1/\beta + s) \leq I$, and $p = -\ln(I/\mu K)/\beta$ otherwise. Put differently, set $p = 1/\beta + s$ unless upon doing so the corresponding demand exceeds I, in which case set price at the higher value of $-\ln(I/\mu K)/\beta$. The latter is also called the stock-clearing price. These arguments are summarized in the following proposition.

Proposition 1 *The optimal riskless price p^0 is given as*

$$p^0 = \max\{1/\beta + s, -\ln(I/\mu K)/\beta\}.$$

It is easy to verify that $\pi^d(p)$ is increasing in I for each p, and that p^0 is decreasing in β and I.

2.2 Single Markdown Opportunity With Stochastic Demand

In this subsection we consider the stochastic version of the above problem. The parameter $z(p) = I/y(p)$ is completely determined by p. However, we suppress its argument p for notational compactness and note that any function of z is an implicit function of p. The total revenue for a realization of ξ can be written as follows:

$$\pi^s(p) = \begin{cases} pD(p) + s(I - D(p)) & \text{if } \xi \leq z, \\ pI & \text{otherwise.} \end{cases} \tag{3}$$

Setting $I = y(p)z$ in (3), taking expectations over all values of ξ, and simplifying, we obtain the following expected revenue function:

$$E[\pi^s(p)] = sI + y(p)(p - s)(\mu - \Theta(z)), \tag{4}$$

where $\Theta(z) = E(\xi - z)^+ = \int_z^\infty [u - z]dF(u)$. It is a common practice to write the expected revenue function as the sum of two functions: a riskless revenue function and a loss function which is caused by demand uncertainty. That approach leads to the following representation: $E[\pi^s(p)] = \pi^d(p) - L(p)$, where $\pi^d(p)$ is as given in (1) and $L(p) = y(p)(p - s)\Theta(z)$ is the expected loss due to demand uncertainty.

Applying Leibnitz's rule to differentiate (4), we obtain our first major result (proof is shown in Appendix A).

Theorem 1 *The optimal price under stochastic demand p^* is obtained by solving the following equation:*

$$p^* = \frac{\nu(z^*)}{\beta} + s, \tag{5}$$

where $\nu(z) = [\mu - \Theta(z)]/[\mu - \Theta(z) - z(1 - F(z))] \geq 1$, and $z^* = I/y(p^*)$.

An immediate consequence of Theorem 1 is that $p^* \geq p^o$. This inequality can be explained by noting that the variance of demand is decreasing in p whereas its coefficient of variation is independent of

p. In fact, the optimal price in a stochastic demand environment exceeds the optimal riskless price in all multiplicative demand models (see Petruzzi and Dada, 1999 for details). Since z is a function of p, the optimal price is obtained via a numerical solution of equation 5. Substituting p^* into (4), we obtain the expected revenue for the optimal price:

$$E[\pi^s(p^*)] = \frac{\nu(z^*)y(p^*)}{\beta}[\mu - \Theta(z^*)] + sI \tag{6}$$

Analogous to the deterministic demand model, we can show that $E[\pi^s(p)]$ is increasing in I, and that p^* is decreasing in I and β. These results are satisfying on an intuitive level since we expect that a larger amount of surplus stock I, and/or less price-sensitive demand (larger β), should be reasons for setting a lower price. That $E[\pi^s(p)]$ is increasing in I can be seen directly by examining (4). In order to verify that p^* is decreasing in I and β, recall first that p and z are related through the relationship $z = \frac{I}{y(p)}$, and that $\frac{dz}{dI} = \frac{1}{y(p)} > 0$. Similarly $\frac{dz}{d\beta} = \frac{pI}{y(p)} = pz > 0$. Next, we use $F^{(1)}(z)$ to denote $\int_0^z F(u)du$, and then differentiate $\nu(z)$ with respect to z, which results in the following observations.

$$\begin{aligned}
\frac{d\nu(z)}{dz} &= \frac{z[F(z)(1-F(z)) - zf(z)] + F^{(1)}(z)[zf(z) - (1-F(z))]}{\left(zF(z) - F^{(1)}(z)\right)^2} \\
&\leq \frac{z[F(z)(1-F(z)) - zf(z)] + zF(z)[zf(z) - (1-F(z))]}{\left(zF(z) - F^{(1)}(z)\right)^2} \\
&= z^2 f(z)[F(z) - 1] \\
&\leq 0.
\end{aligned} \tag{7}$$

The first inequality in (7) follows from the fact that $F(u)$ is an increasing function of u, for all $u \geq 0$. Therefore $F^{(1)}(z) \leq zF(z)$ for all $z \geq 0$. Finally, $\frac{dp^*}{dI} = \frac{z^*}{I\beta}\left(\frac{d\nu(z)}{dz}\,|_{z=z^*}\right) \leq 0$, and a similar series of steps imply that $\frac{dp^*}{d\beta} \leq 0$.

At this stage, it is useful to summarize insights resulting from the analysis of the single markdown problem. These help guide our treatment in the next section. We observe that the optimal price is higher in the stochastic demand model. Also, for each price p, $\pi^d(p) \geq E[\pi^s(p)]$. In particular, this holds for $p = p^*$, which leads to the following conclusion: $E[\pi^s(p^0)] \leq E[\pi^s(p^*)] \leq \pi^d(p^*) \leq \pi^d(p^0)$. Another important observation is that $\pi^d(p)$ and $E[\pi^s(p)]$ are increasing in I, whereas p^0 and p^* are decreasing in both I and β.

3 N – Markdown Opportunities

We now consider the situation in which there are N opportunities to set prices for clearance items, once at the start of each of N periods. We use subscript n to represent the n^{th} period, where $n = 1, 2, \cdots, N$. Thus, the surplus inventory at the start of the n^{th} period in the selling season is denoted by I_n. Bold font is used for vectors. For example $\mathbf{p} = (p_1, \cdots, p_N)$ represents the vector of prices, and $\mathbf{d} = (d_1, \cdots, d_N)$ the vector of demands. In practice N is small, with usually no more than 2 or 3 opportunities to adjust prices. Furthermore, since the entire selling period is relatively small (typically 4-10 weeks), we do not discount revenues from different periods.

3.1 N – Markdown Opportunities With Deterministic Demand

Analogous to (1), the n^{th} period revenue function, which is denoted by ψ_n, can be written as follows:

$$\psi_n(p_n) = \begin{cases} p_n \min\{I_n, d_n\} & \text{if } n < N, \\ \\ p_N \min\{I_N, d_N\} + s_N(I_N - d_N)^+ & \text{otherwise.} \end{cases} \tag{8}$$

where $I_n = (I_{n-1} - d_{n-1})^+$, for each $n \geq 2$, $d_n = \mu_n y_n(p_n)$, $y_n(p_n) = K_n e^{-\beta_n p_n}$, and s_N is the salvage value. Summing both sides of (8) for all n from 1 through N, and simplifying, we obtain the following expression for $\pi^d(\mathbf{p})$:

$$\pi^d(\mathbf{p}) = p_1 I_1 - \sum_{n=1}^{N} (p_n - p_{n+1})(I_n - d_n)^+, \tag{9}$$

where $p_{N+1} \equiv s_N$. Note that $(I_n - d_n)^+$ is zero if $I_1 < \sum_{i=1}^n d_i$, and $(I_1 - \sum_{i=1}^n d_i)$ otherwise. Therefore (9) can be written alternatively as follows:

$$\pi^d(\mathbf{p}) = p_1 I_1 - \sum_{n=1}^{N} (p_n - p_{n+1})(I_1 - \sum_{i=1}^{n} d_i)^+, \tag{10}$$

Let $\mathbf{p}^0 = (p_1^0, \cdots, p_N^0)$ be the optimal price vector for the deterministic problem, i.e., $\mathbf{p}^0 = \arg\{\max_{\mathbf{p}} \pi^d\}$. Solving for $\mathbf{p}^0$ is made difficult by the presence of the $(I_1 - \sum_{i=1}^n d_i)^+$ terms. However, the problem becomes trivial in one special case. Suppose there is ample inventory to meet demand in each period, irrespective of the prices chosen. In that case, the problem of setting prices decouples into N separate one-period problems. From Proposition 1, the corresponding optimal period-n price is $p_n = \frac{1}{\beta_n} + s_N$. Clearly, this solution is optimal overall if $\sum_{i=1}^N \mu_i y_i(\frac{1}{\beta_i} + s_N) \leq I_1$. That is, when the preceeding inequality holds, we have the optimal price vector without further effort. Therefore, in the remainder of this subsection, we assume that $\sum_{i=1}^{N} \mu_i y_i (\frac{1}{\beta_i} + s_N) > I_1$. Our approach is to prove certain properties of the optimal price vector that lead to quick and efficient method for solving the problem.

Observation 1 *For each period n, $1 \leq n \leq N$, the price p_n should be chosen so that either $d_n = I_n$, or $d_n < I_n$. In other words, p_n should never be chosen so as to make $d_n > I_n$.*

It is easy to verify that the above observation must hold. Consider the situation in which $I_n > 0$ and the price in period n is chosen to make $d_n > I_n$. In every such case, the retailer can increase revenue by keeping prices in period 1 through $n-1$ unchanged and simply increasing price in period n until $d_n = I_n$. Notice that since $I_j = 0$ for all $j > n$, periods $n+1$ through N do not affect revenue. An immediate consequence of this observation is that we can impose the following structure on the choice of optimal price vector:

Proposition 2 *The price vector $\mathbf{p}^0$ must be such that one of the following is true:*

i. For some n, where $1 \leq n \leq N$:

$\sum_{j=1}^{n} d_j = I_1$, $I_j = I_1 - \sum_{i=1}^{j-1} d_i > 0$ *for* $j = 1, 2, \cdots, n$, *and* $I_j = 0$, *for all* $j > n$.

ii. $I_1 > \sum_{j=1}^{N} d_j$.

Thus, in order to find $\mathbf{p}^0$, we need to solve at most N deterministic problems such that in the n^{th} problem, period n is the first period in which $d_n = I_n$. For the n^{th} problem, we need to find $n-1$ optimal prices. The price in period n is determined uniquely by the fact that $\sum_{j=1}^{n} d_j = I_1$ and prices in periods $n+1$ through N do not matter since $I_j = 0$ for all $j > n$. For simplicity, we set these to zero. The n^{th} deterministic problem, with objective function π_n^d, can be written as follows:

$$\text{Maximize } \pi_n^d(\mathbf{p}) = p_1 I_1 - \sum_{j=1}^{n-1} (p_j - p_{j+1})(I_1 - \sum_{i=1}^{j} d_i), \tag{11}$$

Subject to:

$$\sum_{i=1}^{j} d_i - I_1 \leq 0, \quad \text{for all } j = 1, 2, \cdots, n-1, \text{ and,} \tag{12}$$

$$\sum_{j=1}^{n} d_j = I_1. \tag{13}$$

We further simplify the solution procedure by utilizing the following result with proof shown in Appendix B.

Proposition 3 *Let $\mathbf{p}_n^0$ denote the solution to the n^{th} problem described in (11) – (13). Then, $\pi^d(\mathbf{p}_N^0) \geq \pi^d(\mathbf{p}_n^0)$ for all n.*

That is, of the N possible cases, $N-1$ are dominated, leaving exactly one candidate. The problem formulated in (11) – (13) can be solved as explained below.

Theorem 2 *The optimal price vector for N^{th} deterministic problem can be obtained as follows:*

$$p_N : I_1 = \sum_{j=1}^{N} \mu_j K_j e^{-\beta_j p_j}, \tag{14}$$

$$p_n = p_N + \frac{1}{\beta_n} - \frac{1}{\beta_N}, \quad \textit{for all } n = 1, 2, \cdots, N-1. \tag{15}$$

Proof of theorem 2 is presented in Appendix C. An efficient procedure for solving the clearance pricing problem with N markdown opportunities and deterministic demand can now be described as follows. Check first if $\sum_{i=1}^{N} \mu_i y_i (\frac{1}{\beta_i} + s_N) \leq I_1$. If this inequality holds, then set $p_n^0 = \frac{1}{\beta_n} + s_N$ and we are done. If not, then solve a single non-linear equation in one unknown, i.e., solve (14) after substituting from (15) into (14). This can be easily implemented on a spreadsheet. In Figure 1, we show how the optimal price discount in the second period of a two-period problem depends on the ratio of β_1 and β_2 for a representative example. In agreement with intuition, the discount increases sharply as price sensitivity declines, i.e., $\beta_1/\beta_2 \to 0$. We can also show (see Corollary below) that the clearance prices are decreasing in the period index under declining price-sensitivity.

Corollary 1 *If price sensitivity of demand is decreasing in period index ($\beta_j \geq \beta_{j-1}$, for all $j \geq 2$), then the optimal prices are also decreasing, i.e., $p_j^0 \leq p_{j-1}^0$.*

Clearly, the above result holds when $p_j^0 = \frac{1}{\beta_j} + s_N$. If we use Theorem 2 to find optimal prices, then Corollary 1 follows directly from equation (15).

We now consider the situation in which price sensitivity does not vary significantly over time, i.e., $\beta_j \approx \beta$ for all j, but we want to explore multiple opportunities to adjust markdowns. In this case, the optimal price has an explicit expression. If $\sum_{i=1}^{N} \mu_i y_i (\frac{1}{\beta} + s_N) \leq I_1$, then the optimal prices are: $p_j = \frac{1}{\beta} + s_N$ for all j. If not, we then use Theorem 2 and it is easy to verify from (15) that $p_j = p_N$ for all $j = 1, 2, \cdots, N-1$. Clearance price p_N is obtained as follows: $p_N = \frac{1}{\beta}[-\ln(I_1/\sum_{j=1}^{N} \mu_j K_j)]$. Therefore, overall optimum clearance price is $p^0 = \max\{\frac{1}{\beta} + s_N, \frac{1}{\beta}[-\ln(I_1/\sum_{j=1}^{N} \mu_j K_j)]\}$. The significance of this result is two-fold. First it shows that a single-discount pricing policy is optimal even though the scale factor of demand ($\mu_j K_j$) does vary from period to period. Put differently, we have shown that the price-sensitivity parameter β_j is the primary determinant of relative prices in different periods. Second, it can be viewed as a multiperiod analog of Proposition 1 of Section 2. The result is also in agreement with a similar observation made by Gallego and Van Ryzin (1994).

It is sometimes convenient to have a single markdown and keep price unchanged in the remainder of the selling season, even if β_j's vary from one period to the next. When additional markdowns incur the cost of relabeling merchandise with new price stickers, there are additional economic reasons for choosing a single discount strategy. The methodology developed above allows us to quickly compute the magnitude of error that results from a single discount scheme for any given example. The details are as follows. Let the price after exercising the opportunity to markdown once and the corresponding objective function be denoted with a subscript f. That is,

$$\pi_f^d(p_f) = p_f I_1 - (p_f - s_N)(I_1 - \sum_{i=1}^{N} d_i)^+. \tag{16}$$

It is easy to verify that the optimal p_f is the unconstrained single markdown price, if at that price total demand is still smaller than available inventory. If not, then the optimal price is the stock clearing price. The unconstrained single markdown price, and the stock clearing single markdown price are obtained, respectively, by solving the following equations:

$$p : \sum_{i=1}^{N} K_i \mu_i e^{-\beta_i p}[1 - \beta_i(p - s_N)] = 0. \tag{17}$$

$$p : \sum_{i=1}^{N} K_i \mu_i e^{-\beta_i p} = I_1. \tag{18}$$

In Figure 2, we plot the penalty for using the single discount price ($\{\pi^d(\mathbf{p}^0) - \pi_f^d(p_f^*)\} \times 100/\pi^d(\mathbf{p}^0)$), as a function of β_1/β_2 for the same data that is used to generate Figure 1. Note that the penalty for using a single markdown scheme declines both when price sensitivity of demand is relatively constant over time ($\beta_1/\beta_2 \to 1$) and when price sensitivity declines significantly ($\beta_1/\beta_2 \to 0$), achieving its maximum for moderate decline in sensitivity. Retail managers are thus justified in using a single markdown, not only when demand retains about the same sensitivity to price over time, but also when price sensitivity drops sharply in later selling periods.

How do optimal prices vary with I_1 and the demand scale vectors $\boldsymbol{\mu}$ and $\boldsymbol{K}$? We answer this question by considering two cases. If $\sum_{i=1}^{N} \mu_i y_i(\frac{1}{\beta_i} + s_N) \leq I_1$, then the optimal prices are $p_j = \frac{1}{\beta_j} + s_N$, which are unaffected by I_1, $\boldsymbol{\mu}$ and $\boldsymbol{K}$. On the other hand, if $\sum_{i=1}^{N} \mu_i y_i(\frac{1}{\beta_i} + s_N) > I_1$, then from (15) we notice that for each $n < N$ the amount by which p_n exceeds p_N depends only on demand price-sensitivities. Thus, optimal prices in all periods are affected in exactly the same manner as the price in period N. If I_1 increases, p_N needs to decrease in order to maintain the equality in equation (14). Similarly the left hand side of equation (14) is increasing in $\boldsymbol{\mu}$ and $\boldsymbol{K}$, implying that p_N must increase in order to maintain the equality. Overall, it follows that prices are increasing in demand scale ($\boldsymbol{\mu}$ and $\boldsymbol{K}$) and decreasing in the amount of leftover stock at the beginning of the first period (I_1). These observations agree with intuition and with the results reported in Section 2.

3.2 N – Markdown Opportunities With Stochastic Demand

We begin this section by observing that the stochastic N-period clearance pricing problem does not have a myopic optimal solution even when demands in different periods are assumed independent (see Heyman and Sobel 1984, pages 84-85 for conditions under which myopic solutions exist). This means that the problem of setting optimal prices is a difficult problem. Smith and Achabal (1998) note that such pricing decisions would need to be jointly optimized by stochastic dynamic programming and that the state space for this problem is extremely large.

Clearance prices can be chosen either once at the start of the clearance season and not updated during the remainder of the selling season, or they may be dynamically updated based on the realized sales history at each decision epoch. The latter is a particularly hard problem since demands in different periods are likely to be correlated and the decision problem does not have a Markov structure. Therefore in this section, we focus our efforts on deriving structural properties of the problem of choosing clearance prices when prices are chosen once at the start of the markdown period. These are used to develop bounds and a heuristic solution.

Using equation (9), the N-period expected revenue can now be written as follows:

$$E[\pi^s(\mathbf{p})] = p_1 I_1 - E[\sum_{n=1}^{N}(p_n - p_{n+1})(I_n - D_n)^+], \tag{19}$$

where, as before, p_{N+1} equals s_N, $I_n = (I_{n-1} - D_{n-1})^+$, and $D_n = y_n(p_n)\xi_n$. Note that we are using upper case D_n to denote demand to underscore the fact that demand is now assumed random. For each price vector $\mathbf{p}$, the components of demand vector $\mathbf{D}$ may be arbitrarily correlated through the interdependence of the random components ξ_n. Also note that $(I_n - D_n)^+$ is zero if $I_1 < \sum_{i=1}^{n} D_i$

and $(I_1 - \sum_{i=1}^{n} D_i)$ otherwise. Therefore, (19) can be further simplified to the following equivalent form:

$$E[\pi^s(\mathbf{p})] = p_1 I_1 - \sum_{n=1}^{N} (p_n - p_{n+1}) E[(I_1 - \sum_{i=1}^{n} D_i)^+], \tag{20}$$

where our goal is to find a price vector $\mathbf{p}^*$ that maximizes $E[\pi^s(\mathbf{p})]$. Owing to the complicated form of the right hand side of (20), it is difficult to prove concavity of $E[\pi^s(\mathbf{p})]$ in $\mathbf{p}$. Thus, ensuring that a numerical solution procedure identifies the optimal $\mathbf{p}$ is a challenging problem.

Let $\mathbf{x} = (x_1, \cdots, x_N)$ denote a realization of the random components of demand, i.e., x_n is a realization of ξ_n for each n. Let $v_n(\mathbf{p}, \mathbf{x}) = (I_1 - \sum_{i=1}^{n} y_i(p_i) x_i)^+ = [g_n(\mathbf{p}, \mathbf{x})]^+$, where $g_n(\mathbf{p}, \mathbf{x}) = I_1 - \sum_{i=1}^{n} y_i(p_i) x_i$. With these definitions in hand, it is possible to show that for each n and each price vector $\mathbf{p}$, the function $v_n(\mathbf{p}, \mathbf{x})$ is a convex function of $\mathbf{x}$. The function $g_n(\mathbf{p}, \mathbf{x})$ is a linear function of $\mathbf{x}$ and $[u]^+$ is a convex function of u. Thus, v_n is the composition of a convex function with a linear function, which is known to be convex. Next, applying Jensen's inequality, we see that $E(I_1 - \sum_{i=1}^{n} y_i(p_i)\xi_i)^+ \geq (I_1 - \sum_{i=1}^{n} y_i(p_i) E\xi_i)^+$. If we assume decreasing prices, then the following result is straightforward upon setting $d_i = y_i(p_i) E\xi_i$ in the equivalent deterministic problem.

Proposition 4 *For each price vector $\mathbf{p}$ such that $p_n \geq p_{n+1}$, $n = 1, 2, \cdots, N$, $E[\pi^s(\mathbf{p})] \leq \pi^d(\mathbf{p})$. Therefore, the deterministic problem can be used to obtain the following lower and upper bounds on the stochastic solution. Let $\mathbf{p}^*$ be the optimal price vector for the stochastic demand problem within the set of all price vectors with declining prices, then*

$$E[\pi^s(\mathbf{p}^0)] \leq E[\pi^s(\mathbf{p}^*)] \leq \pi^d(\mathbf{p}^*) \leq \pi^d(\mathbf{p}^0). \tag{21}$$

Since $\mathbf{p}^*$ is unknown, $E[\pi^s(\mathbf{p}^0)]$ and $\pi^d(\mathbf{p}^0)$ are the lower and upper bounds (respectively) that we can compute relatively easily. The expected revenue $E[\pi^s(\mathbf{p}^0)]$ can be estimated by sampling from the joint distribution of random components of demand and taking the average of realized values from (20). Note that the assumption of declining prices is not a serious limitation from a practical view point since retailers behave in this way when setting prices for clearance merchandise. Numerical experiments reported in Section 4 show that $\pi^d(\mathbf{p}^0)$ is a relatively loose upper bound. However, $E[\pi^s(\mathbf{p}^0)]$ is quite tight. Thus $\mathbf{p}^0$ is a good heuristic solution. However, we develop a heuristic which is even better, as explained below.

The framework necessary for obtaining the proposed heuristic solution is based on the generalized Jensen bounds discussed in Huang, Ziemba and Ben-Tal (1977). In this method, the support Ξ of the vector $\boldsymbol{\xi} = (\xi_1, \cdots, \xi_N)$ is successively partitioned into finer divisions in order to improve the bound. In our problem setting, this method works as follows.

Let k index the cells of a ν-fold partition of Ξ. In the stochastic programming literature, cells are also called scenarios and we use these terms interchangeably. Cells of this partition are defined such that each B^k is convex, $B^k \neq \emptyset$, $B^k \cap B^\ell = \emptyset$ if $k \neq \ell$, and $\cup_{k=1}^{\nu} B^k = \Xi$. The partition is denoted by $\mathcal{B}^{(\nu)}$ where $\mathcal{B}^{(\nu)} = \{B^k,\ k = 1, ..., \nu\}$. Note that the following rectangular partition, which we use in our numerical experiments, has all of the above-mentioned properties.

$$B^k = [a_1^{(k)}, b_1^{(k)}] \times [a_2^{(k)}, b_2^{(k)}] \times \cdots \times [a_N^{(k)}, b_N^{(k)}], \tag{22}$$

where N is again the number of periods. Let $\gamma^k = P(B^k)$ be the probability on the cell, and $\bar{\boldsymbol{\xi}}^k$ be the vector of conditional expectations of the random demand components on the cell. Specifically, $\bar{\xi}_i^k = \int_{B^k} (x_i/\gamma^k) dF_N(\mathbf{x})$, where $F_N(\mathbf{x})$ is the joint CDF of $\boldsymbol{\xi}$. Having developed these partitions, and noting that $v_n(\mathbf{p}, \mathbf{x})$ is convex, we apply the generalized Jensen bounding method of Huang et al. (1977) [p. 320] to obtain:

$$Ev_n(\mathbf{p}, \mathbf{x}) \geq \sum_{k=1}^{\nu} \gamma^k v_n(\mathbf{x}, \bar{\boldsymbol{\xi}}^k). \tag{23}$$

Successively finer partitions, which can be obtained, for example, by dividing each cell into 2^N cells, lead to larger lower bounds in (23).

Substituting from (23) into (20), we have for each $\mathbf{p}$,

$$\begin{aligned}
E[\pi^s(\mathbf{p})] &\leq p_1 I_1 - \sum_{n=1}^{N} (p_n - p_{n+1}) \sum_{k=1}^{\nu} \gamma^k (I_1 - \sum_{i=1}^{n} y_i(p_i) \bar{\xi}_i^k)^+ && (24)\\
&\equiv \hat{\pi}^s(\mathbf{p}) && (25)\\
&\leq p_1 I_1 - \sum_{n=1}^{N} (p_n - p_{n+1}) (I_1 - \sum_{i=1}^{n} y_i(p_i) \sum_{k=1}^{\nu} \gamma^k \bar{\xi}_i^k)^+ && (26)\\
&= p_1 I_1 - \sum_{n=1}^{N} (p_n - p_{n+1}) (I_1 - \sum_{i=1}^{n} y_i(p_i) E\xi_i)^+ && (27)\\
&= \pi^d(\mathbf{p}) && (28)
\end{aligned}$$

Inequality (26) follows from the fact that for all a_i, $(a_1)^+ + \cdots + (a_N)^+ \geq (a_1 + \cdots + a_N)^+$. Notice the upper bound becomes successively better as ν increases and, in the limit, we obtain the true objective function of the stochastic problem corresponding to any price vector. In reality, the function $\hat{\pi}^s$ stabilizes very quickly as seen in Figure 3, which shows the plot of $\hat{\pi}^s(\mathbf{p}^0)$ as a function of ν. The problem data corresponds to a two-period problem in which the random component of demand has a bivariate normal distribution with a correlation coefficient of 0.5. Parameters of demand distribution are chosen such that the chance of a negative demand is almost zero. Negative demands (customer returns) are not uncommon when dealing with clearance merchandise, but they tend to be insignificant overall. Notice that when ν exceeds about 50, there is little marginal gain from increasing the number of partitions.

From now on, we shall assume that ν has been chosen appropriately so that the function $E[\pi^s(\mathbf{p})] \approx \hat{\pi}^s(\mathbf{p})$ for all $\mathbf{p}$. Our efforts are therefore directed at optimizing the function $\hat{\pi}^s$. The advantage of using the partitioning scheme is that it allows us to quickly compute $\hat{\pi}^s$ for any given price vector, which is necessary for generating and refining heuristic solutions. For each price vector $\mathbf{p}$, let $k^+_{\mathbf{p}}(n)$ be the subset of cells/scenarios for which there is some inventory at the start of the n^{th} period, i.e.,

$$k^+_{\mathbf{p}}(n) = \{j : \sum_{i=1}^{n-1} y_i(p_i)\bar{\xi}_i^j < I_1\} \tag{29}$$

If $k^+_{\mathbf{p}}(n)$ is non-null, we further divide it into the following subsets:

$$k^=_{\mathbf{p}}(n) = \{j : \sum_{i=1}^{n} y_i(p_i)\bar{\xi}_i^j = I_1\} \tag{30}$$

$$k^>_{\mathbf{p}}(n) = \{j : \sum_{i=1}^{n} y_i(p_i)\bar{\xi}_i^j > I_1\} \tag{31}$$

$$k^<_{\mathbf{p}}(n) = \{j : \sum_{i=1}^{n} y_i(p_i)\bar{\xi}_i^j < I_1\} \tag{32}$$

That is, $k^=_{\mathbf{p}}(n) \cup k^>_{\mathbf{p}}(n)$ is the subset of scenarios under which all stock available at the beginning of period n is sold in that period.

Proposition 5 *Price vectors for which $k^+_{\mathbf{p}}(n) \neq \emptyset$, but the set $k^<_{\mathbf{p}}(n) \cup k^=_{\mathbf{p}}(n)$ is empty, are dominated. This holds for any period index $n = 1, \cdots, N$.*

Proof: Let $\mathbf{p}$ be a price vector such that $k^+_{\mathbf{p}}(n) \neq \emptyset$ but the set $k^<_{\mathbf{p}}(n) \cup k^=_{\mathbf{p}}(n)$ is empty. This means that price in period n is chosen such that demand exceeds available inventory under each scenario. Note that prices in period $n+1, \cdots, N$ do not matter since there is nothing left to sell in those periods under any scenario. Consider what will happen if we increase p_n while keeping $p_j,\ j \neq n$ fixed. It is easy to see that increasing p_n so long as $\sum_{i=1}^{n} y_i(p_i)\bar{\xi}_i^k \geq I_1$ for all k, and $\sum_{i=1}^{n} y_i(p_i)\bar{\xi}_i^k = I_1$ for at least one k, strictly increases expected revenue in period n, while leaving revenue in other periods unchanged. Thus, for every price vector with the property mentioned above, it is easy to find an alternate price vector that yields higher overall expected revenue. Hence proved. #

Proposition 6 *Price vectors for which $k^{<}_{\mathbf{p}}(n) = \emptyset$ for at least one $n < N$ are dominated.*

Proof: The above result is the deterministic analog of Proposition 2. It implies a price vector that results in the sale of all inventory in $n < N$ periods under all scenarios is dominated. The proof is realized by showing that for every price vector $\mathbf{p}$ that clears all stock in $n < N$ periods under all scenarios, there exists another price vector $\mathbf{p}'$ with higher expected revenue. From proposition 5, the set $k^{<}_{\mathbf{p}}(n) \cup k^{=}_{\mathbf{p}}(n)$ cannot be empty. Furthermore, if $k^{<}_{\mathbf{p}}(n)$ is non-empty, then we are done since there is at least one scenario under which we have something to sell in period $n+1$. Thus, from now onward, we assume that only $k^{=}_{\mathbf{p}}(n)$ and $k^{>}_{\mathbf{p}}(n)$ are non-empty.

In order to show that the price vector $\mathbf{p}$ is dominated, we now construct another price vector $\mathbf{p}'$ such that $p_j = p'_j$ for all $j \neq n, n+1$ and $p'_{n+1} = p_n$. We also set p'_n just slightly more than p_n. By making the increase small enough, we can ensure that whatever inventory is not sold in period n for scenarios in $k^{=}_{\mathbf{p}}(n)$, can be completely sold in period $n+1$ and that demand under scenarios $k^{>}_{\mathbf{p}}(n)$ does not become smaller than available inventory at the start of period n. This is always possible since the demand is strictly positive for all price in the range $[0, \infty)$ under all scenarios. More formally, $\mathbf{p}'$ is chosen such that

$$\begin{aligned} \hat{\pi}^s(\mathbf{p}') &= \sum_{i=1}^{n-1}\sum_{k=1}^{\nu} p_i \min\{(I_1 - \sum_{r=1}^{i-1} y_r(p_r)\bar{\xi}^k_r)^+, y_i(p_i)\bar{\xi}^k_i\}\gamma^k && (33)\\ &+ \sum_{k\in k^{>}_{\mathbf{p}}(n)} p'_n(I_1 - \sum_{i=1}^{n-1} y_i(p_i)\bar{\xi}^k_i)\gamma^k && (34)\\ &+ \sum_{k\in k^{=}_{\mathbf{p}}(n)} \{p'_n y_n(p'_n)\bar{\xi}^k_n + p_n(I_1 - \sum_{i=1}^{n} y_i(p'_i)\bar{\xi}^k_i\}\gamma^k && (35)\\ &> \hat{\pi}^s(\mathbf{p}). && (36) \end{aligned}$$

The last inequality follows from the fact that $p'_n > p_n$, $p'_{n+1} = p_n$, and the sum of inventory sold in periods n and $n+1$ is identical under the two price vectors. Hence proved. #

Propositions 5 and 6 imply that the optimal price vector is such that either there is inventory leftover under all scenarios at the end of period N (recall this is called the unconstrained solution), or prices are chosen such that $\sum_{i=1}^{N} y_i(p_i)\bar{\xi}^k_i = I_1$ for at least one scenario $k = 1, \cdots, \nu$. Finding the optimal price vector when $\sum_{i=1}^{N} y_i(p_i)\bar{\xi}_i^k = I_1$ for some k continues to be a hard problem. We still need to consider the aggregate impact on revenues of a candidate price vector under all scenarios. However, if we focus only on scenario k for which $\sum_{i=1}^{N} y_i(p_i)\bar{\xi}_i^k = I_1$, and ignore all other scenarios, we then recognize that this problem has the solution already obtained in Theorem 2. On the other hand, if there is ample stock, equation (20) reduces to $E[\pi^s(\mathbf{p})] = p_1 I_1 - \sum_{n=1}^{N}(p_n - p_{n+1})(I_1 - \sum_{i=1}^{n} y_i(p_i)E(\xi_i))$, and $p_i = 1/\beta_i + S_N$ for all i. Our overall heuristic approach can now be summarized as follows:

i. Set $p_i = 1/\beta_i + s_N$ for all i. If with these prices, $\sum_{i=1}^{N} y_i(p_i)\bar{\xi}_i^k \leq I_1$ for all k, then these are optimal prices and we are done.

ii. If step 1 does not yield a solution, set $\sum_{i=1}^{N} y_i(p_i^k)\bar{\xi}_i^k = I_1$ and solve for the optimal price for each $k = 1, \cdots, \nu$ using relationship (15). This will generate price vectors $\mathbf{p}^k$, $k = 1, \cdots, \nu$ and as many possible values of $\hat{\pi}^s$. Choose the price vector that maximizes $\hat{\pi}^s$ as the heuristic solution, i.e., $\hat{\mathbf{p}}^* = \arg\{\max_{\mathbf{p}^k} \hat{\pi}^s(\mathbf{p}^k)\}$.

In order to test the accuracy of the heuristic suggested above, we obtain the optimal clearance prices and corresponding expected revenue via an exhaustive search. Computational effort associated with the search procedure can be greatly reduced by establishing upper and lower bounds on optimal prices, which are obtained as explained below.

Let p_n^* denote the optimal price in period n. Then, it can be argued that the only reason to not choose $p_n^* = 1/\beta_n + s_N$ is that there is insufficient inventory to meet the resulting demand under at least one scenario. Such inventory constraint restricts sales thereby providing the incentive to raise prices, or else properties established in either Proposition 5 or Proposition 6 no longer hold. This implies that $p_n^* \geq 1/\beta_n + s_N$. The function $\hat{\pi}^s(\mathbf{p})$ can be seen as a convex combination of realized revenues under ν scenarios, where the revenue under scenario k is as given in (10) after substituting $\boldsymbol{\mu} = \bar{\boldsymbol{\xi}}^k$. We know that prices in the deterministic problem are higher when demand scale is larger. Therefore, if we replace each $\bar{\boldsymbol{\xi}}^k$ by $\max_k\{\bar{\boldsymbol{\xi}}^k\}$, for each k, the solution to the resulting deterministic problem is an upper bound on the possible values of p_i^*. Let this price vector be denoted by $\boldsymbol{p}^{\max}$, then from above arguments we have established that $1/\beta_n + s_N \leq p_n^* \leq p_n^{\max}$.

Analogous to the deterministic demand case, we can also use the model above to determine the optimal single markdown price. It is obtained as follows:

$$p_f^* = \arg\{\max_p [pI_1 - (p - s_N) \sum_{k=1}^{\nu} \gamma^k (I_1 - \sum_{i=1}^{N} y_i(p)\bar{\xi}_i^k)^+]\}. \tag{37}$$

Using arguments similar to those presented in the previous Section, it can be proved that if $I_1 > \sum_{i=1}^N y_i(p)\bar{\xi}_i^k$ for all k, then p_f^* is obtained from solving equation (17). Since $p : \sum_{i=1}^N y_i(p)\bar{\xi}_i^k > I_1$ for all k cannot be optimal, this leaves N cases where in the k^{th} case $\sum_{i=1}^N y_i(p)\bar{\xi}_i^k = I_1$. Each such case results in a non-linear equation in one unknown similar to equation (18). The optimal single discount price is the price that results in the overall maximum expected revenue.

4 Examples and Insights

In this section, we report results of numerical experiments with two markdown opportunities (i.e., $N = 2$). We use the multivariate normal distribution to represent the random components of demand. Figure 4 shows the upper and lower bounds, the heuristic and optimal solutions, and the best single discount price solution as a function of the ratio β_1/β_2. The two solid lines are the upper and lower bounds, i.e., $\pi^d(\mathbf{p}^0)$ and $\hat{\pi}^s(\mathbf{p}^0)$ respectively. The dashed line shows the optimal solution found via an exhaustive search, the dash-dotted line shows the heuristic solution, and the dotted line shows the solution obtained after using the optimal single discount price. In terms of our notation, the last three lines represent, respectively the quantities $\hat{\pi}^s(\mathbf{p}^*)$, $\hat{\pi}^s(\hat{\mathbf{p}}^*)$, and $\hat{\pi}^s(p_f^*)$. We also calculated the performance of the different bounds. As evident $\pi^d(\mathbf{p}^0)$ is a loose upper bound and on average it is about 14.1% larger than the heuristic solution (maximum deviation 15.7% and minimum deviation 9.9%). The optimal solution and the heuristic solution are quite close with an average gap of 1.4% (maximum gap 1.9% and minimum gap 0.24%). In fact for β_1/β_2 greater than 0.4, the optimal and the heuristic solutions are indistinguishable. The penalty for using a single discount pricing scheme varies from 0.064% to 4.56% of the heuristic solution, with an average error of 1.58%. Figure 5 shows the different error calculations. The solid line shows the gap between $\pi^d(\mathbf{p}^0)$ and the heuristic solution, the dash-dotted line shows the difference between the optimal and the heuristic solution, and the dotted line shows the penalty due to a single discount price scheme.

We also investigated the effect of demand correlation on the goodness of the heuristic solution. The results are plotted in Figure 6. It shows the values of $\pi^d(\mathbf{p}^0)$ (top solid line), $\hat{\pi}^s(\mathbf{p}^*)$ (dashed line), $\hat{\pi}^s(\hat{\mathbf{p}}^*)$, and $\hat{\pi}^s(\mathbf{p}^0)$. Notice that the last two values are too close and appear as a single solid line at the bottom. The problem data has $I_1 = 1000$, $s_2 = 0.1$, $K_1 = K_2 = 1000$, $\beta_1 = 1$, $\mu_1 = 4$, $\sigma_1 = 1.25$, $\beta_2 = 2.0$, $\mu_2 = 3$, $\sigma_2 = 1.0$, and ρ is varied from -0.99 to 0.96 in steps of 0.1. The difference between the heuristic solution and the optimal solution is small with an average gap of about 0.5% (maximum 1.1%, minimum 0.41%). Increasing positive correlation tends to lower expected profits.

Two points are worth noting in Figure 4. First, we see that the solution of the equivalent deterministic problem is a reasonable heuristic. It is easy to implement on a spreadsheet and fast solution can be obtained. Second, it appears that the single discount pricing scheme underperforms the solution obtained from an equivalent deterministic problem. As a practical matter, this means that retail managers will do better to solve the (faster/easier) equivalent deterministic model with time-dependent price sensitivities, than use a single markdown strategy. This is particularly true when price sensitivities drop only moderately in later periods.

5 Impact of Demand Variability

Retailers often use advertisement to affect sales of seasonal items. They also use market research tools in an attempt to improve demand forecasts. It therefore seems useful to identify circumstances under which efforts to affect the joint distribution of demand are beneficial. In this Section we show that the expected revenue is ordered opposite to the convex ordering of partial sums of demands. This means that promotional efforts that reduce variability of demand are beneficial. The significance of this result is that it holds for arbitrary demand distributions in different periods that have an arbitrary correlation structure. In order to prove this result, we need additional notation and definitions, which are provided next.

If X and Y are arbitrary random variables, and $E[f(X)] \le E[f(Y)]$ for all convex functions $f : \mathcal{R} \to \mathcal{R}$ for which expectations exist, then $X \le_{cx} Y$ (see, for example, Shaked and Shanthikumar, 1996, pp. 56). It follows that if $X \le_{cx} Y$, then $E(X) = E(Y)$ and $\mathrm{Var}(X) \le \mathrm{Var}(Y)$ (Shaked and Shanthikumar, 1994; Chapter 2). With these definitions in hand, and recalling that for each $\mathbf{p}$, $f_n(\mathbf{p}, \mathbf{x})$ is a convex function of $\mathbf{x}$, it is immediately obvious that if $\sum_{i=1}^n \xi_i \le_{cx} \sum_{i=1}^n \xi_i'$, then $E(I_1 - \sum_{i=1}^n D_i)^+ \le E(I_1 - \sum_{i=1}^n D_i')^+$. If we further assume declining prices, i.e., $p_n \ge p_{n+1}$, for all $n = 1, \cdots, N-1$, this observation immediately leads us to Proposition 7 below.

Proposition 7 *For each declining price vector* $\mathbf{p}$*, if* $\sum_{i=1}^{n} \xi_i \leq_{cx} \sum_{i=1}^{n} \xi'_i$ *for each* n*, then* $E[\pi^s(\mathbf{p})] \geq E[\pi^{s\prime}(\mathbf{p})]$.

Proof of Proposition 7 follows directly from the observations made in the paragraph above and from (20).

For those more familiar with multivariate convex orders, we point out that the convex ordering of partial sums is related to (but weaker than) the *positive-linear-combination-convex order* (*plcx* in short), which in turn is weaker than the multivariate analog of the convex order defined above. [If $\mathbf{X}$ and $\mathbf{Y}$ are arbitrary n-dimensional random vectors, $n \geq 1$, and $E[f(\mathbf{X})] \leq E[f(\mathbf{Y})]$ for all convex functions $f : \mathcal{R}^n \to \mathcal{R}$ for which expectations exist, then $\mathbf{X} \leq_{cx} \mathbf{Y}$ (see, for example, Shaked and Shanthikumar, 1996, Chapter 4).] Given two N-dimensional vectors $\mathbf{X}$ and $\mathbf{Y}$, we say that $\mathbf{X}$ is smaller than $\mathbf{Y}$ in the *plcx* order, denoted as $\mathbf{X} \leq_{plcx} \mathbf{Y}$, if for all convex functions $\phi : \mathcal{R} \to \mathcal{R}$, $E\phi(a_1X_1 + \cdots + a_NX_N) \leq E\phi(a_1Y_1 + \cdots + a_NY_N)$ with $a_i \geq 0$, when the expectations exist. Thus, if $\boldsymbol{\xi} \leq_{plcx} \boldsymbol{\xi'}$, then $\sum_{i=1}^{n} D_i \leq_{cx} \sum_{i=1}^{n} D_i$ follows upon setting $\mathbf{a} = (y_1(p_1), \cdots, y_n(p_n), 0, \cdots, 0)$, where exactly first n components are positive and the rest are 0.

6 Summary and Conclusions

Nearly all research on clearance pricing has assumed that the price-sensitivity of demand is time-invariant. However, our discussions with managers at several major retailers have convinced us that this assumption is unrealistic. Customers are clearly less likely to buy fashion items, even with deep discounts, towards the end of the selling period. Given that price markdowns for clearance items are made at multiple times during the clearance period and that price-sensitivity changes over time, it stands to reason that the optimal prices should also change over the clearance period. Models are needed for finding the optimal prices over a multiple-markdown clearance period with changing demand parameters. This paper presents deterministic and stochastic models for both single and multiple markdown clearance pricing problems.

The models provide several important managerial insights which are summarized next. If the price-sensitivity parameter does not change during the clearance period (in our notation, this means that $\beta_j \approx \beta$), then problem essentially becomes a one-markdown problem and the optimal prices do not change during the clearance period. Similarly, if the price-sensitivity parameters change dramatically during the clearance period (that is, $\beta_1 \gg \beta_2$), then the optimal solution is to attempt to clear nearly all of the inventory in the first period, and only first markdown price matters. In this case also, the problem becomes essentially a one-markdown problem. However, if the price-sensitivity parameters change moderately during the clearance period (a realistic scenario), the optimal prices will not be uniform over the clearance period and the penalty of applying a single markdown and keeping the price unchanged thereafter is significant.

Our empirical analysis suggests that it is generally better to apply a multiple-markdown deterministic model to determine clearance prices than a stochastic single markdown model. This is good news due to the fact that the solution method for the single and multiple markdown stochastic models require complex numerical methods, whereas the solution method proposed here for the multiple-markdown deterministic problem is exact, efficient, and quite straightforward to implement. Deterministic model is also less data-hungry as retailers need only estimate the average of the random component of demand in each period.

There are two results of theoretical interest presented in this paper. We obtain upper and lower bounds for the expected revenue and the optimal prices for the stochastic multiple-markdown problem. We also show that efforts to affect the joint distribution of demand are beneficial when they result in convexly decreasing partial sums of demand. This claim stands irrespective of the nature of demand correlations across different periods.

This paper helps to underscore the fact that the problem of finding optimal prices when demand is random and arbitrarily correlated, and its price sensitivity parameter is time-dependent, is a difficult problem. Additional research is needed to improve our understanding of the stochastic dynamic clearance pricing problem with the features mentioned above. To our knowledge, that problem has not been studied.

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Appendix

A Proof of Theorem 1

Upon differentiating (4), we obtain the first-order optimality equation as follows:

$$\begin{aligned}\frac{dE[\pi^s(p)]}{dp} &= y(p)[\mu-\Theta(z)]\left[1-\beta(p-s)\left(1-\frac{z[1-F(z)]}{\mu-\Theta(z)}\right)\right] \\ &= y(p)[\mu-\Theta(z)]\left[1-\frac{\beta(p-s)}{\nu(z)}\right]=0. \qquad \text{(A.1)}\end{aligned}$$

Since $z=I/y(p)$, and $\Theta(z)=E(\epsilon-z)^+$, it implies as $p\to\infty$, $y(p)\to 0$, and $\Theta(z)\to 0$. Therefore, the optimality equation (A.1) yields two roots: either $p=\infty$, or $p=p^*$ that satisfies the following equation:

$$\frac{\beta(p-s)}{\nu(z)}=1. \qquad \text{(A.2)}$$

Furthermore, p^* is a unique solution in the range (s,∞). This is seen by taking the derivative of the left-hand side (A.2) which yields:

$$\frac{d[\beta(p-s)/\nu(z)]}{dp}=\beta\Big(\frac{\nu(z)-(p-s)\nu'(z)}{\nu(z)^2}\Big)>0. \qquad \text{(A.3)}$$

In the above inequality, which follows from (7), the prime notation signifies derivative with respect to p. Recall that in (7) we proved $\nu'(z)=(d\nu(z)/dz)(dz/dp)\le 0$, for all z. Therefore, when $p>s$, the left-hand side of (A.2) is monotone increasing in p, resulting in a unique point at which the equality holds.

Using (4) and taking the limit as p approaches infinity, we observe that $\lim_{p\to\infty}E[\pi^s(p)]=sI$. However, for any $p\ge s$, we know from the same expression that $E[\pi^s(p)]\ge sI$. The logic behind this simple on an intuitive level: since we can sell the any amount of surplus stock at salvage s, any price greater than s should result in at least as much revenue as the lower bounding salvage-value based revenue. Technically, the proof follows from observing that $\mu-\Theta(z)\ge 0$ for all p. Since $p^*>s$, we see that the solution in (A.2) dominates the option to set an extremely high price. That is, we need only consider the solution in (A.2). It remains to show that the solution to (A.2) is indeed a maximum. In order to do that, we take the second derivative of (4) to obtain:

$$\frac{\partial^2E(\pi^s(p))}{\partial p^2}\,|_{p=p^*}=\delta(p^*)[\nu'(z^*)-\beta]\le 0, \qquad \text{(A.4)}$$

where we have used $\delta(p) = [\mu - \Theta(z)]y(p)/\nu(z) \geq 0$ for notational compactness. The above inequality also follows from the fact that $\nu'(z) = (d\nu(z)/dz)(dz/dp) \leq 0$, for all z. This is true in particular at $z = z^*$. The inequality (A.4) now results from noting that $\delta(p) \geq 0$ for all p.

In summary, we have proved that p^* is a local maximum and that the objective function value at p^* is at least as much as the value at $p = \infty$. Since there are only two roots of the first order optimality equation, p^* is indeed a unique maximum. #

B Proof of Proposition 3

Let $\mathbf{p}^k = (p_1^k, \cdots, p_k^k, \cdot, \cdots, \cdot)$ denote a feasible price vector for the k^{th} problem, where $k < N$. Notice that prices in periods $k+1$ through N can be set arbitrarily and have no impact on π_k^d and that $p_j^k < \infty$ for each $j \leq k$. In order to prove Proposition 3, we will show that for every $\mathbf{p}^k$, there exists a price vector $\mathbf{p}^{k+1}$ for the $(k+1)$-period problem such that $\pi_{k+1}^d(\mathbf{p}^{k+1}) \geq \pi_k^d(\mathbf{p}^k)$. Since this inequality holds for every $k < N$, and in particular when $\mathbf{p}^k$ is set equal to $\mathbf{p}_k^0$, a repeated application of the inequality leads to the conclusion reported in Proposition 3.

Let d_j^r denote the demand in period j of the r-period problem. Then, vectors $\mathbf{p}^k$ and $\mathbf{p}^{k+1}$ are feasible if and only if the corresponding demands satisfy the following equality: $\sum_{j=1}^k d_j^k = \sum_{j=1}^{k+1} d_j^{k+1} = I_1$. We construct the vector $\mathbf{p}^{k+1}$ such that $p_j^{k+1} = p_j^k$ for every $j = 1, \cdots, k-1$, and $p_{k+1}^{k+1} = p_k^k$. Prices in periods $k+2$ through N do not matter. Notice that $d_j^k = d_j^{k+1}$ for every $j < k$, and that d_{k+1}^{k+1} is strictly positive, but no more than d_k^k (due to decreasing price sensitivity), for the chosen price $p_{k+1}^{k+1} = p_k^k$. Now, we adjust the period-k price of the $(k+1)$-period problem until the amount demanded in that period is exactly equal to $d_k^k - d_{k+1}^{k+1}$. Since this quantity is strictly less than d_k^k, it follows that $p_k^{k+1} > p_k^k$. The total revenue in the $(k+1)$-period problem is then:

$$\begin{aligned}
\pi_{k+1}^d(\mathbf{p}^{k+1}) &= \sum_{j=1}^{k+1} d_j^{k+1} p_j^{k+1} \\
&= \sum_{j=1}^{k-1} d_j^k p_j^k + p_k^{k+1}[d_k^k - d_{k+1}^{k+1}] + p_k^k d_{k+1}^{k+1} \\
&= \sum_{j=1}^{k} d_j^k p_j^k + [p_k^{k+1} - p_k^k][d_k^k - d_{k+1}^{k+1}] \\
&\geq \pi_k^d(\mathbf{p}^k) \qquad \text{(B.5)}
\end{aligned}$$

The last inequality follows from the fact that $p_k^{k+1} > p_k^k$, and $d_k^k \geq d_{k+1}^{k+1}$.

C Proof of Theorem 2

The following proof is presented for a k-period problem, for any k such that $k = 1, \cdots, N$. The results reported in Theorem 2 are realized by setting $k = N$.

Let $L_k = p_1 I_1 - \sum_{n=1}^{k-1}[p_n - p_{n+1}](I_1 - \sum_{i=1}^n d_i) + \sum_{n=1}^k \gamma_n[(\sum_{i=1}^n d_i) - I_1]$, where multipliers γ_n are non-negative for $n = 1, 2, \cdots, k-1$ and γ_k is unrestricted. Using standard Kuhn-Tucker conditions for nonlinear optimization (see, for example, Luenberger (1984), pages 314-317) the optimal price vector must satisfy the following first order necessary conditions:

$$\frac{\partial L_k}{\partial p_j} = 0 \quad \text{for all } j = 1, 2, \cdots, k, \tag{C.1}$$

$$\gamma_n([(\sum_{i=1}^{n} d_i) - I_1]) = 0, \quad \text{for all } n = 1, 2, \cdots, k, \text{ and,} \tag{C.2}$$

$$\sum_{n=1}^{k} d_n = I_1. \tag{C.3}$$

Thus, we have $2k$ simultaneous equations in as many unknowns. From their definition, we know that $\partial d_n / \partial p_n = -\beta_n d_n = -\beta_n K_n \mu_n e^{-\beta_n p_n}$. Using this relationship and differentiating L_k, we obtain the following equations.

$$\frac{\partial L_k}{\partial p_j} = d_j \left(1 - \sum_{n=j}^{k-1} [p_n - p_{n+1}]\beta_j - \sum_{n=j}^{k} \gamma_n \beta_j \right), \tag{C.4}$$

where $j = 1, 2, 3, \cdots, k-1$, and

$$\frac{\partial L_k}{\partial p_k} = (I_1 - \sum_{n=1}^{k-1} d_n) - \gamma_k \beta_k d_k \tag{C.5}$$

From the fact that $\sum_{n=1}^{j} d_n$ must be strictly less than I_1 for all $j = 1, 2, \cdots k-1$, and complimentary slackness conditions (see equations C.2) it follows that $\gamma_j = 0$ for all $j = 1, 2, \cdots k-1$. Substituting from equation (C.3) in equation (C.5) and setting the derivative to zero we obtain

$$(1 - \gamma_k \beta_k) d_k = 0 \tag{C.6}$$

from where it immediately follows that $\gamma_k = 1/\beta_k$. Recall that $d_k = K_k \mu_k e^{-\beta_k p_k}$ and it is strictly positive for all values of $p_k \in [0, \infty)$. Using arguments similar to those presented for the deterministic one-period problem it can be confirmed that setting $p_j = \infty$, for $j = 1, 2, \cdots, k$ results in a local minimum and hence we do not consider that possibility further.

Substituting the values of γ_n in (C.4), and setting the derivatives to zero, we obtain the following first order necessary conditions, one for each $j = 1, 2, \cdots, k-1$:

$$d_j \left(1 - \sum_{n=j}^{k-1} [p_n - p_{n+1}]\beta_j - \beta_j / \beta_k \right) = 0. \tag{C.7}$$

For $p_j \in [0, \infty)$, the above is equivalent to

$$\sum_{n=j}^{k-1} [p_n - p_{n+1}]\beta_j = 1 - \frac{\beta_j}{\beta_k}, \tag{C.8}$$

which further simplifies to

$$p_j = p_k + \frac{1}{\beta_j} - \frac{1}{\beta_k}. \tag{C.9}$$

Substituting for p_j's from above in (C.3), we obtain

$$\sum_{j=1}^{k} K_j e^{-\beta_j[p_k + \frac{1}{\beta_j} - \frac{1}{\beta_k}]} = I_1. \tag{C.10}$$

It is easy to see that the left hand side of equation (C.10) is strictly decreasing in p_k. Since, demand is at least as much as inventory at price $p_k = s_N$, there is a unique solution of (C.10) such that $p_k \in [s_N, \infty)$. That is, equations (C.9) and (C.10) can be used to determine p_j's uniquely for each $j = 1, 2, \cdots, k$. Let this clearance price vector be denoted by $\mathbf{p}_k^0$.

The unique solution to the first order necessary conditions is a maximum (and therefore a global maximum) if $-L_k$ is positive semidefinite at $\mathbf{p}_k^0$. To verify that we take second partial derivatives of $-L_k$ and find that

$$\frac{\partial(-L_k)}{\partial p_j} = -d_j \left(1 - \beta_j(p_j - p_k) - \sum_{n=j}^{k} \gamma_n \beta_j\right). \tag{C.11}$$

$$\frac{\partial^2(-L_k)}{\partial p_j^2} = \beta_j d_j \left(2 - \beta_j(p_j - p_k) - \sum_{n=j}^{k} \gamma_n \beta_j\right). \tag{C.12}$$

$$\frac{\partial^2(-L_k)}{p_a p_b} = 0, \quad \text{for all } a \neq b, \text{ and } 1 \leq a, b \leq k-1. \tag{C.13}$$

Clearly, $\frac{\partial^2(-L_k)}{\partial p_j^2} |_{\mathbf{p}_k^0} = \beta_j d_j \geq 0$. It can now be verified that the determinants of the $\ell \times \ell$ principal minor of the $(k-1) \times (k-1)$ Hessian matrix of the second derivatives of $-L_k$ is positive for all $\ell = 1, 2, \cdots, k-1$. Therefore, $-L_k$ is positive semi-definite at $\mathbf{p}_k^0$. Hence proved. #

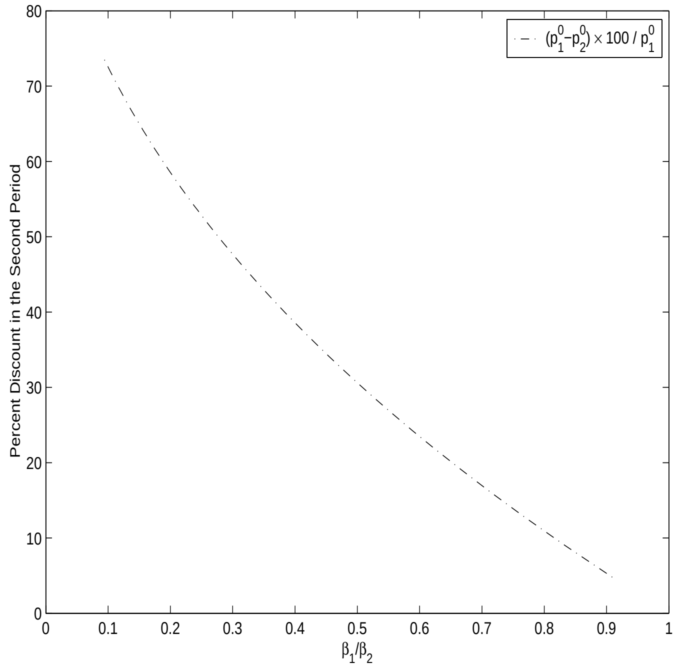

Figure 1: Optimal percent discount in period two of a two-period problem. Data: $I_1 = 1000$, $s_2 = 0.1$, $K_1 = 1000$, $\beta_1 = 1$, $\mu_1 = 3$, $K_2 = 1000$, $\mu_2 = 4$, and β_2 is varied from 1.1 to 11 in steps of 0.1.

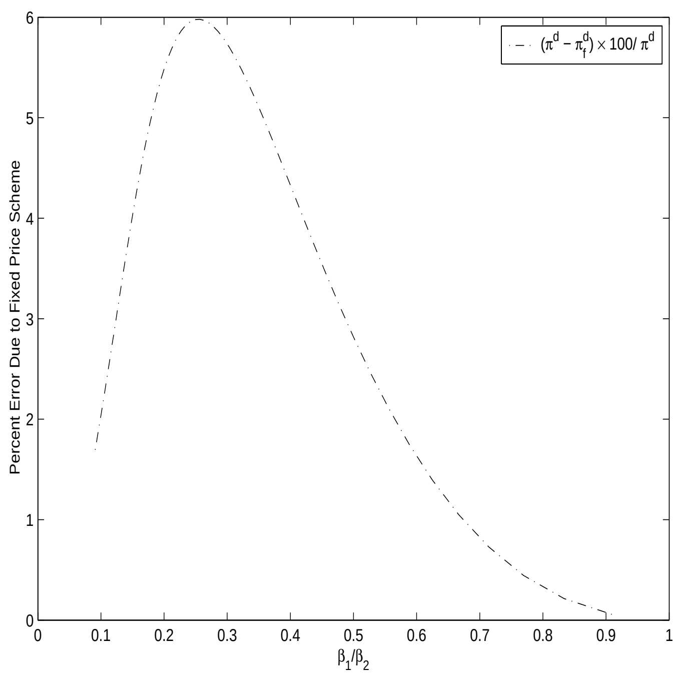

Figure 2: Percent penalty for using the single discount price scheme. Problem data are as follows: $I_1 = 1000$, $s_2 = 0.1$, $K_1 = 1000$, $\beta_1 = 1$, $\mu_1 = 3$, $K_2 = 1000$, $\mu_2 = 4$, and β_2 is varied from 1.1 to 11 in steps of 0.1.

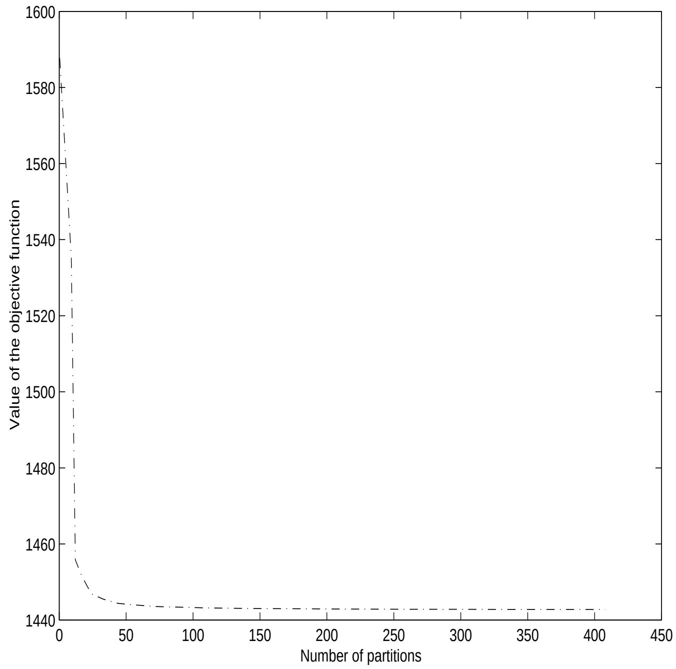

Figure 3: Value of $\hat{\pi}^s(\mathbf{p}^0)$ as a function of the number of partitions ν in a two-period problem. Problem data are: $I_1 = 1000$, $s_2 = 0.1$, $K_1 = 1000$, $K_2 = 1000$, $\beta_1 = 1$, $\beta_2 = 1.5$, $\boldsymbol{\xi}$ is bivariate normal with parameters $\mu_1 = 3$, $\sigma_1 = 1$, $\mu_2 = 4$, $\sigma_2 = 1.25$, and $\rho = 0.5$.

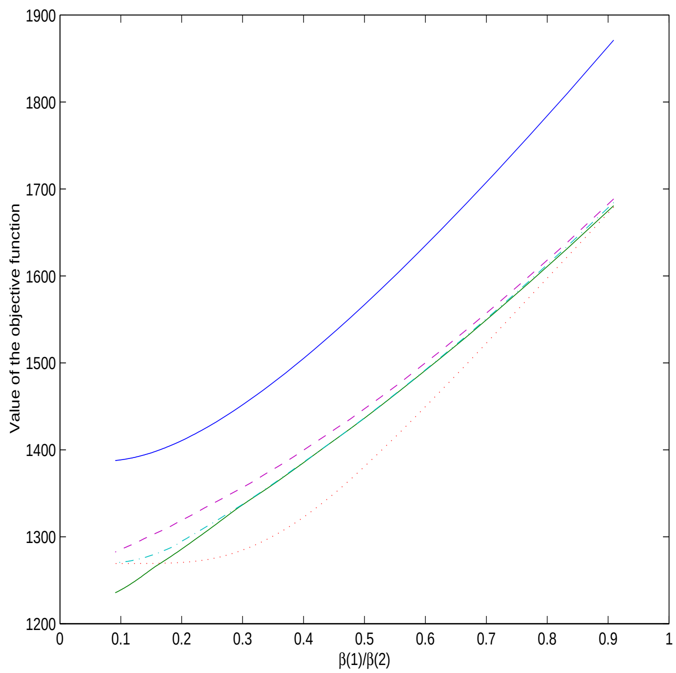

Figure 4: Plot of $\pi^d(\mathbf{p}^0)$, $\hat{\pi}^s(\mathbf{p}^*)$, $\hat{\pi}^s(\hat{\mathbf{p}}^*)$, $\hat{\pi}^s(p_f^*)$, and $\hat{\pi}^s(\mathbf{p}^0)$ as a function of β_1/β_2. Problem data are: $I_1 = I_2 = 1000$, $K_1 = K_2 = 1000$, $\beta_1 = 1$, $\mu_1 = 4$, $\sigma_1 = 1.25$, $\mu_2 = 3$, $\sigma_2 = 1$, $\rho = 0.5$, and $s_2 = 0.1$.

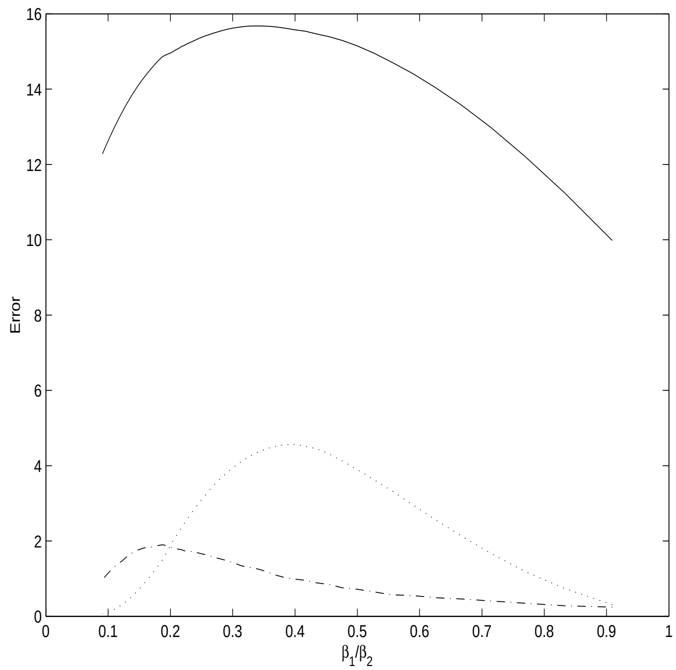

Figure 5: Plot of error as a function of β_1/β_2. Problem data are the same as in Figure 4.

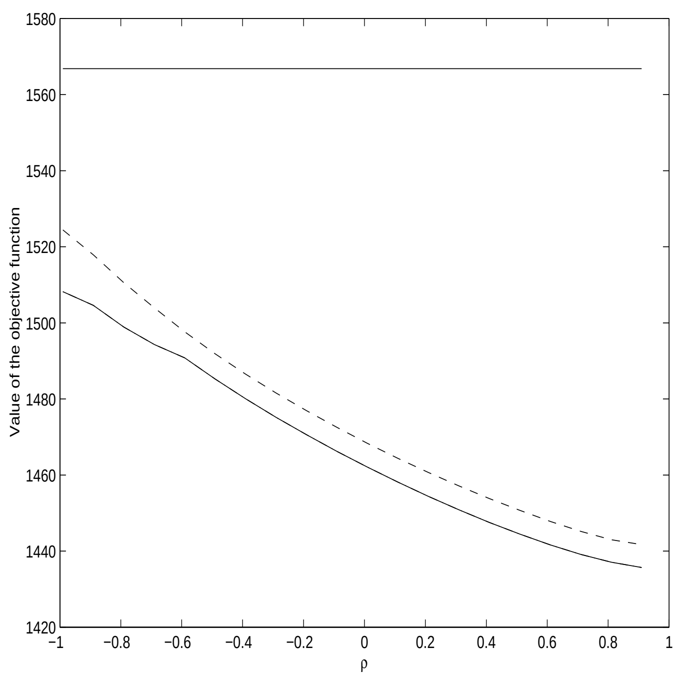

Figure 6: Plot of $\pi^d(\mathbf{p}^0)$ (top solid line), $\hat{\pi}^s(\mathbf{p}^*)$ (dashed line), $\hat{\pi}^s(\hat{\mathbf{p}}^*)$, and $\hat{\pi}^s(\mathbf{p}^0)$ as a function of the correlation coefficient ρ.